

RADIATIVE-CONVECTIVE PROCESSES AND CHANGES OF THE FLUX OF THERMAL RADIATION INTO SPACE WITH INCREASING OPTICAL THICKNESS OF THE ATMOSPHERE

V. G. Gorshkov, A. M. Makarieva, T. Pujol¹

¹Department of Physics, Universitat de Girona, Girona, Catalonia, Spain

A b s t r a c t

It is shown that at large values of atmospheric optical thickness the account of non-radiative convective processes of energy transfer in the lower atmosphere does not violate the validity of Eddington's approximation for description of radiative transfer in the atmosphere. Non-radiative convective fluxes in the atmosphere propagate till a finite height, which is independent of atmospheric optical thickness at large values of the latter. Relative contribution of the dissipative power of non-radiative convective energy fluxes into formation of the outgoing flux of thermal radiation diminishes linearly with growing optical thickness of the atmosphere.

РАДИАЦИОННО-КОНВЕКТИВНЫЕ ПРОЦЕССЫ И ИЗМЕНЕНИЕ ПОТОКА ТЕПЛОВОГО ИЗЛУЧЕНИЯ В КОСМОС С УВЕЛИЧЕНИЕМ ОПТИЧЕСКОЙ ТОЛЩИНЫ АТМОСФЕРЫ

В. Г. Горшков, А. М. Макарьева, Т. Пуйоль¹

¹Отделение физики, Университет Жероны, Жерона, Каталония, Испания

А н н о т а ц и я

Показано, что при больших значениях оптической толщины атмосферы учет нерадиационных конвективных потоков энергии не нарушает справедливости приближения Эддингтона для всей атмосферы. Высота распространения нерадиационных конвективных потоков энергии в атмосфере стремится к конечному пределу при увеличении оптической толщины атмосферы. Вклад мощности диссипации нерадиационных конвективных потоков в формирование уходящей в космос энергии теплового излучения линейно уменьшается с ростом оптической толщины атмосферы.

1 Introduction

Investigating the nature of thermal stability of the Earth's climate, currently exposed to globally significant anthropogenic forcing, represents an important problem. Will the increasing atmospheric CO_2 concentration drive the climate to an elevated yet stable value of global mean surface temperature, or there is a probability of climate stability broken and terrestrial climate driven to a state unsuitable for life existence. Scientific answers to these questions are strongly dependent on how the outgoing flux of thermal radiation into space changes in response to changes in atmospheric concentrations of greenhouse substances.

Greenhouse substances in the atmosphere (among which water vapour, clouds and carbon dioxide are the major ones) are responsible for resonance absorption of thermal radiation emitted from the Earth's surface. They further re-emit it approximately isotropically into the upper and lower hemispheres, thus returning some part of thermal radiation back to the surface. Thus, the upward flux of thermal radiation $F^+(z)$ [W m^{-2}] is maximum at the surface and decreases with atmospheric altitude z , coinciding with the absorbed flux of solar radiation F_{in} outside the atmosphere, $F_e \equiv F^+(z=\infty) = F_{\text{in}}$. At the Earth's surface the upward flux of thermal radiation $F_s^+ \equiv F^+(z=0)$ is related to surface temperature T_s by the well-known Stephen-Boltzmann formula for the black body, $F_s^+ = \sigma T_s^4$. The increase of the upward flux of thermal radiation at the surface F_s^+ as compared to the outgoing flux of thermal radiation into space F_e quantifies the magnitude of the greenhouse effect.

The number of molecules of a given greenhouse substance in the atmospheric column of an area equal to the cross-section of absorption of thermal radiation by this greenhouse substance is called the optical thickness of the atmosphere τ_s . It is equal to the total number in the atmosphere of layers of this greenhouse substance that are distanced from each other by average free path length of thermal photons. Optical depth τ at height z is determined as the number of such layers above z :

$$\tau \equiv \int_z^{\infty} \frac{dz'}{l(z')}, \quad \tau_s \equiv \int_0^{\infty} \frac{dz'}{l(z')}, \quad (1.1)$$

where $l(z)$ is the average free path length of thermal photons at height z , $l(z) \equiv (\Sigma N(z))^{-1}$, Σ is the molar cross-section of absorption of thermal photons by one mole of the greenhouse substance ($\Sigma \equiv \sigma_s A$, where σ_s is the molecular cross-section, A is the Avogadro number), $N(z)$ is the molar concentration of this substance at height z . Thermal radiation escaping into space originates predominantly from the upper radiative layer of the atmosphere corresponding to optical depth $\tau \sim 1$, which is distanced from the outer space by one free path length of thermal photons.

In the absence of non-radiative energy fluxes in the atmosphere (the so-called state of radiative equilibrium) the ratio F_s^+ / F_e , i.e. the magnitude of the greenhouse effect, is linearly dependent on the optical thickness of the atmosphere τ_s :

$$F_s^+ / F_e = 1 + k\tau_s, \quad (1.2)$$

where k is an approximately constant magnitude, which changes by no more than 15% with τ_s changing from zero to infinity. In particular, for $\tau_s > 1$ we have $k = 3/4$. In such a case (1.2) corresponds to the well-known Eddington's approximation [1,2]. In Section 2 we present a somewhat different from conventional derivation of Eq. (1.2) from the radiative transfer equation.

In accordance with Clausius-Clapeyron equation, the amount of water vapour in the atmosphere grows exponentially with increasing surface temperature, rising approximately twofold per each ten degrees of surface temperature increment. Taking into account the related increase in cloudiness, which absorbs thermal radiation rather evenly over the whole thermal spectrum, this corresponds to an exponential decline of the outgoing flux of thermal radiation F_e and further heating of the planet's surface at $F_e < F_{in}$. Such a situation is commonly referred to as a runaway greenhouse effect. The described positive feedback between the rise in surface temperature and atmospheric water content may lead to the loss of thermal stability of climate.

In the atmosphere the absorbed solar energy generates non-radiative convective fluxes of latent and sensible energy, which could in principle significantly perturb the above radiative equilibrium pattern of atmospheric energy transfer. While propagating in the atmosphere, these non-radiative fluxes undergo dissipation and ultimately convert into outgoing thermal radiation. The dissipative power of non-radiative energy fluxes in both stationary and non-stationary cases may not exceed the power of absorbed solar radiation F_{in} . In the modern atmosphere the dissipative power of non-radiative fluxes of latent and sensible energy does not exceed 27% of F_{in} [3,4].

One can imagine a situation when a certain finite amount of energy of the non-radiative fluxes dissipates in the upper radiative layer of the atmosphere $\tau \sim 1$ irrespective of the magnitude of the atmospheric optical thickness τ_s . If this were be the case, the outgoing flux of thermal radiation into space F_e would not diminish exponentially with growing T_s but reached a finite plateau determined by the value of non-radiative dissipative power at $\tau \sim 1$. In such a case it would be possible to discuss climate change in terms of transitions between different stable values of surface temperature, stable shift of climatic zones etc. At any value of surface temperature T_s the outgoing flux of thermal radiation into space would be equal to sum of the radiative component decreasing with τ_s in accordance with Eddington's approximation (1.2) and constant convective component representing dissipation of dynamic energy into thermal radiation at the upper radiative layer $\tau \sim 1$. The latter would be complementing the former, so that their sum could be made equal to the absorbed flux of thermal radiation F_{in} .

In Section 3 we show that in a gaseous atmosphere in a gravitational field the height z_c of convective zone is independent of τ_s at $\tau_s \gg 1$. This means that the optical depth corresponding to this height, $\tau_c(z_c)$, increases proportionally to atmospheric optical thickness, $\tau_c(z_c) \propto \tau_s$ at $\tau_s \rightarrow \infty$. With increasing surface temperature T_s the ratio F_s^+ / F_{in} of the upward flux of thermal radiation at the surface to the absorbed flux of solar radiation grow as well. The dissipative power of convective energy fluxes F_c , which is limited by the absorbed flux of solar radiation F_{in} , $F_c < F_{in}$, does not surpass optical depth τ_c , which increases with growing τ_s . This means that the ratio of radiative to non-radiative energy fluxes F^+ / F_c within the convective zone increases proportionally to τ_s . The perturbation of radiative equilibrium by non-radiative energy fluxes diminishes and the upward flux of thermal radiation in the atmosphere determined by Eddington's approximation becomes the dominant form of energy transport in the atmosphere. The outgoing flux of thermal radiation tends therefore to zero with increasing τ_s and does not reach a constant plateau.

The above statements are based on analysis of general properties of radiative transfer in a gaseous atmosphere in the gravitational field, the details of which are given in Section 3. At any height z the upward flux of thermal radiation F^+ in a given interval of wave lengths can be related to the so-called brightness temperature T_b , which is equal to temperature of a black body radiator emitting the same flux of radiation F^+ within this interval. Brightness temperature characterises energy density of thermal radiation and the population number of the excited levels of the absorption bands of greenhouse substances, which is in thermal equilibrium with radiation at large values of optical depth τ . At the Earth's surface the brightness temperature of all radiation frequencies approximately coincides with surface temperature T_s , as thermal radiation of the Earth's surface is close to that of a black body. As well as the upward flux of thermal radiation F^+ , brightness temperature T_b drops with height, so that there appears a negative vertical gradient of brightness temperature $0 > dT_b / dz \equiv -G_b$. Everywhere below we use the notion of lapse rate to denote the absolute magnitude of negative temperature gradients.

Inelastic collisions of air molecules with molecules of greenhouse substances bring about additional non-radiative non-resonance excitation of energy levels of the absorption bands of the greenhouse substances, which further relax to the ground state emitting thermal photons. Thus, the energy of thermal motion of air molecules escapes into radiation and, ultimately, into space. As a result, the air temperature drops and a negative vertical temperature gradient $0 > dT / dz \equiv -G$ is formed in the atmosphere. At the Earth's surface both temperatures, T_b and T_s , coincide with temperature of the Earth's surface T_s , which is the single scale of all temperatures in the atmosphere.

Brightness temperature T_b is a function of optical depth τ and atmospheric optical thickness τ_s . According to Eddington's approximation, at large values of $\tau_s \geq \tau \gg 1$

brightness temperature becomes the function of ratio τ/τ_s only. The condition of hydrostatic equilibrium in the well-mixed atmosphere of the Earth's type corresponds to the equality $\tau/\tau_s = p/p_s$, where p/p_s is the ratio of atmospheric pressure at optical depth τ and at the surface. This ratio is a function of ratio z/h_s , where $h_s = RT_s/Mg$ is the height of a homogeneous atmosphere representing the single scale of height z in the considered problem, R is the gas constant, M is the molar mass of air, g is the acceleration of gravity. We thus have $T_b = T_s f(z/h_s)$. The lapse rate of brightness temperature becomes $G_b = (T_s/h_s) f'(z/h_s)$, where $T_s/h_s = Mg/R$ is a τ_s - and T_s -independent magnitude.

Due to collisional conversion of non-radiative to radiative energy in the lower atmosphere, the lapse rate of air temperature G may be close to the lapse rate of brightness temperature G_b and never exceeds it. Convective energy fluxes exist in that part of the atmosphere $0 \leq z \leq z_c$, where G_b is larger than some threshold value Γ corresponding to the advent of convection. The threshold value of $\Gamma \equiv (T_s/h_s)\alpha$, $\alpha = \text{const}$, is determined by the same scale T_s/h_s as G_b [5,6]. As far as function $f(x)$ and its derivative $f'(x)$ both decrease monotonously with x , convective processes are confined to the height interval $0 \leq z \leq z_c$, where the height of convective zone z_c is determined from equation $f'(z_c/h_s) = \alpha$. It yields $z_c = h_s \varphi(\alpha)$, where $\varphi(x)$ is the reverse function of $f'(x)$. Thus, the ratio z_c/h_s is a magnitude independent of atmospheric optical thickness τ_s , while optical depth τ_c corresponding to z_c grows linearly with τ_s . Therefore, with increasing optical thickness of the atmosphere, $\tau_s \rightarrow \infty$, the outgoing flux of thermal radiation into space F_e changes in accordance with Eddington's approximation (1.2) and may not reach a finite plateau. In Sections 3 and 4 we show that the results obtained are valid for both non-condensable greenhouse substances (like CO_2) with constant mass fraction, as well as for condensable greenhouse substances (water vapour), with mass fraction strongly dependent on height z .

In an isothermal atmosphere water vapour can be saturated only immediately above the water surface, $z=0$, where its concentration is determined by the Clausius-Clapeyron equation. Water vapour concentration drops exponentially with height, so that vapour becomes unsaturated. However, in the presence of an air temperature lapse rate G exceeding certain critical value Γ_L prescribed by molecular properties of water, water vapour may remain saturated in the entire atmospheric column. In such a case the decrease of water vapour concentration with height is no longer governed by the equation of hydrostatic equilibrium, but is determined by how rapidly air temperature drops with height.

As already noted, thermal radiation into space is predominantly emitted from the upper radiative layer $\tau \sim 1$. In accordance with definition of optical depth τ (1.1), the amount of water vapour in this layer is constant and independent of the magnitudes of upward thermal radiation flux at the surface F_s^+ and surface temperature T_s . On the

other hand, the amount of saturated water vapour is uniquely related to air temperature. Consequently, constant amount of water vapour in the layer $\tau \sim 1$ corresponds to constant air temperature $T_1 \equiv T(\tau = 1)$ of this layer. Under the assumption of local thermodynamic equilibrium for this layer, the outgoing flux of thermal radiation within the absorption interval of water vapour $F_{\text{H}_2\text{O}}$ equals $\sigma T_1^4 \delta_{\text{H}_2\text{O}}$, where $\delta_{\text{H}_2\text{O}}$ is the share of thermal flux within the absorption interval of water vapour. As far as T_1 is constant, this flux remains constant as well and does not depend on surface concentration of water vapour which grows exponentially with increasing surface temperature T_s . This result, obtained in a number of radiative-convective models, see, e.g. [2,7], stands in contradiction with the above general conclusion on exponential drop of $F_{\text{H}_2\text{O}}$ with increasing T_s (Section 3). The source of this contradiction is the assumption of local thermodynamic equilibrium, which, as discussed in Section 4, appears unjustified for the upper atmosphere, including $\tau \sim 1$, at large τ_s .

We sum up that under conditions similar to terrestrial ones in that there are convective fluxes of energy, non-condensable (CO_2) and condensable (H_2O) greenhouse substance present in the atmosphere and an infinite liquid hydrosphere at its base, the ratio between the outgoing flux of thermal radiation into space F_e to the upward flux of thermal radiation at the Earth's surface F_s^+ decreases inversely proportionally to the atmospheric optical thickness τ_s at $\tau_s \gg 1$. Taking into account the exponential rise of the atmospheric content of water vapour with increasing surface temperature and the related increase in atmospheric cloudiness which "closes" the atmospheric windows, we conclude that the absolute value of the outgoing flux of thermal radiation F_e drops exponentially with increasing surface temperature leading, in the absence of independent negative feedback processes, to the runaway greenhouse effect.

2 Radiative transfer equation and Eddington's approximation

In the planar three-dimensional case, which corresponds to global averaging over longitude and latitude, the equation for transfer of radiation of a given wave length can be written as [1,2]:

$$\mu \frac{\partial I(\mu, z)}{\partial z} = -\frac{1}{l(z)} I(\mu, z) + \frac{S(z)}{l(z)}. \quad (2.1)$$

Here $I(\mu, z)$ is the intensity of radiation, defined as the energy transported per unit time in a given direction through unit perpendicular area at height z by a bundle of rays propagating within a unit solid angle, when the solid angle where measurements are taken tends to zero [8], dimension $[\text{W m}^{-2} \text{sr}^{-1}]$; $l(z)$, as before, see (1.1), is the average free path length of photons; μ is the cosine between the given direction and the vertical axis z ; $S(z)/l(z)$ is the density of radiative power per unit volume. Function $S(z)$, called source function, is assumed to be isotropic, i.e. μ -independent.

The first term in the right-hand side of (2.1) describes absorption of radiation in the corresponding volume, the second one describes emission of radiation within the same volume. Eq. (2.1) represents the law of energy conservation: change in the energy of the ray, $\partial I(\mu, z)$, as the ray travels path $\partial s = \partial z / \mu$, is equal to the difference between the amounts of radiation absorbed and emitted by the matter along that path. Note that the source function $S(z)$ accounts both for radiation emitted due to resonance excitation of the radiating molecules during their interaction with thermal photons, as well as for radiation emitted due to non-resonance collisional excitation of the radiating molecules.

According to (2.1), in free space, where both absorption ($I(z) = \infty$) and emission ($S(z) = 0$) are absent, the intensity should remain constant along the path of any ray. This is indeed the case if the intensity $I(\mu, z)$ is defined, as above, as the mathematical limit for an infinitely small solid angle ([8], p. 72). If the intensity is defined for any finite unit of solid angle (see, e.g., [2], p. 16), Eq. (2.1) does not hold for free space. (This can be easily shown on the example of the point source of radiation, for which the flux of energy through a finite solid angle decreases as the bundle of emitted rays diverges in space. Due to this fact planets located at a different distance from the Sun receive different insulation fluxes.) As far as all physical measurements are characterised by finite sensitivity, the intensity (2.1), defined for an infinitely small solid angle, appears to be a variable that cannot be measured directly. Measurable values are those obtained by integrating (2.1) over solid angle, which, in the planar case considered, corresponds to integrating over μ .

Changing variables in (2.1) from altitude z to optical depth τ as prescribed by (1.1) we obtain the following conventional form of the radiative transfer equation (2.1):

$$\mu \frac{\partial I(\mu, \tau)}{\partial \tau} = I(\mu, \tau) - S(\tau), \quad (2.2)$$

$$S(\tau) = I(0, \tau). \quad (2.3)$$

Eq. (2.3) is obtained when one puts $\mu = 0$ in (2.2).

Integrating (2.2) over μ we use the traditional convenient notations [1]:

$$J(\tau) \equiv J^+(\tau) + J^-(\tau), \quad H(\tau) \equiv H^+(\tau) - H^-(\tau), \quad (2.4)$$

$$J^+(\tau) \equiv \frac{1}{2} \int_0^1 I(\mu, \tau) d\mu, \quad J^-(\tau) = \frac{1}{2} \int_{-1}^0 I(\mu, \tau) d\mu, \quad (2.5)$$

$$H^+(\tau) = \frac{1}{2} \int_0^1 \mu I(\mu, \tau) d\mu, \quad H^-(\tau) = -\frac{1}{2} \int_{-1}^0 \mu I(\mu, \tau) d\mu. \quad (2.6)$$

Variables $J(\tau)$ and $H(\tau)$ correspond to energy density $E = 4\pi J / c$ [J m^{-3}] and energy flux $F = 4\pi H$ [W m^{-2}], where c is velocity of light. Superscripts + and - refer to radiation propagating into the upper and lower hemispheres, respectively.

Calculating τ -derivatives of both sides of all the relations in (2.5) and (2.6) and using (2.2) we obtain:

$$\frac{dH^{\pm}(\tau)}{d\tau} = \pm \left[J^{\pm}(\tau) - \frac{1}{2} S(\tau) \right], \quad (2.7)$$

$$\frac{dH(\tau)}{d\tau} = J(\tau) - S(\tau), \quad (2.8)$$

$$\frac{dJ(\tau)}{d\tau} = \frac{1}{2} \int_{-1}^1 \frac{d\mu}{\mu} (I(\mu, \tau) - S(\tau)). \quad (2.9)$$

Equation (2.9) shows that the important relation (2.3) is necessary for the measurable value of $dJ(\tau)/d\tau$ to be finite. If relation (2.3) does not hold, the τ -derivative of radiative energy density would be infinite at all τ , which is physically implausible. This stresses that source function $S(z)$ is prescribed by the value of intensity $I(0, \tau)$, see (2.3), and cannot be introduced as an independent external field.

We now consider the case of radiative equilibrium, when the non-radiative fluxes of excitation of the greenhouse substance molecules are absent, and the source function is determined by resonance radiative excitation only. In such a case all radiation emitted equals all radiation absorbed, so that the following equalities hold:

$$S(\tau) = J(\tau), \quad (2.10)$$

$$\frac{dH(\tau)}{d\tau} = 0, \quad H(\tau) = H(\tau = 0) \equiv H = \text{const}, \quad (2.11)$$

$$\frac{dH^{\pm}(\tau)}{d\tau} = \frac{1}{2} [J^{+}(\tau) - J^{-}(\tau)]. \quad (2.12)$$

The constant value of H (2.11) represents the net flux of thermal radiation, which coincides with the outgoing flux of thermal radiation at $\tau = 0$. This corresponds to the following boundary conditions [1,9]:

$$H^{+}(0) = H, \quad H^{-}(0) = 0, \quad J^{+}(0) = \sqrt{3}H, \quad J^{-}(0) = 0. \quad (2.13)$$

The flux of resonance radiative excitation of molecules increases with growing τ due to the increasing energy density of thermal radiation. One can assume therefore that at large values of $\tau \gg 1$ the intensity $I(\mu, \tau)$ becomes an almost isotropic function only weakly depending on μ [1]. This allows one to replace $I(\mu, \tau)$ for $I(0, \tau) = J(\tau)$, see (2.3), in all integrals of (2.5) and (2.6). Then, performing integration over μ in (2.5), we obtain:

$$J^{\pm}(\tau) = \frac{J(\tau)}{2}; \quad H^{\pm}(\tau) = \frac{1}{2} J^{\pm}(\tau). \quad (2.14)$$

Now, we multiply Eq. (2.2) by μ and integrate both parts of this equation over μ , putting $I(\mu, \tau) = I(0, \tau) = J(\tau)$ in the left-hand side of (2.2) but using definition

$H \equiv \frac{1}{2} \int_{-1}^1 \mu I(\mu, \tau) d\mu$ in the right-hand side of (2.2) (this integral equals zero at

$I(\mu, \tau) = I(0, \tau)$). We then obtain $(1/3)dJ(\tau)/d\tau = H$ and $J(\tau) = 3H\tau + c_E$. Finally, for the asymptote $f(\tau) \equiv H^{+}(\tau)/H = H^{+}(\tau)/H^{+}(0)$ we have $f(\tau) = (3/4)\tau + c_H$ at $\tau \gg 1$.

The constant $c_H \neq c_E/4$, as far as relation $H^{+}(\tau) = (1/4)J(\tau)$ is only valid for the lead-

ing, i.e. τ -linear, term. At small $\tau \rightarrow 0$ we have $f(\tau \rightarrow 0) = 1$. Thus, for all values of τ one can put approximately

$$f(\tau) = 1 + k\tau, \quad k = \frac{3}{4}. \quad (2.15)$$

Relation (2.15) exactly corresponds to Eddington's approximation, which assumes $c_\varepsilon = 2$ [1]. One can directly check it putting $J(\tau) = J_\varepsilon(\tau) = 3H(\tau + 2/3)$, which is the common form of Eddington's approximation (see [1], Eq. (82.15)), into the integral of Schwarzschild-Milne type defining $H^+(\tau)$ in terms of $J(\tau)$ [1,9]:

$$H^+(\tau) \equiv \frac{1}{2} \int_0^1 \mu d\mu \int_0^\infty e^{-x} J(\tau + \mu x) dx. \quad (2.16)$$

Hopf [10] obtained the constant term in the asymptotic expression for energy density $J(\tau)$ at $\tau \rightarrow \infty$: $J(\tau) = 3H(\tau + q(\tau))$, where $q(\tau)$ is the so-called Hopf function, $q(\tau) = 0.710 + o((1/\tau) \ln \tau)$ at $\tau \rightarrow \infty$. Putting the asymptotic expression $J(\tau) = 3H(\tau + 0.710)$ into (2.16) one obtains for the asymptote of $H^+(\tau)$ at $\tau \gg 1$ that $H^+(\tau) = H((3/4)\tau + 1.033)$, $c_H = 1.033$.

Formally, Eddington's approximation (2.15) is valid for $\tau \gg 1$ and $\tau \rightarrow 0$. However, at $\tau \gg 1$ the constant term in (2.15) differs from the asymptotic constant $c_H = 1.033$ by only 3.3%. On the other hand, at $\tau \ll 1$ one has from (2.12) and (2.13) $f(\tau) = 1 + (3/4)^{1/2} \tau$ [1,9] and the linear term in (2.15) appears to be proportional to $(3/4)^{1/2}$ rather than to Eddington's $3/4$. However, transition between the two approximations, from $k = (3/4)^{1/2}$ to $k = 3/4$ (2.15), occurs approximately at $\tau \sim 0.3$ [9], where the difference between $f(\tau)$ values obtained from the two approximations does not exceed several per cent. At smaller τ the difference between the two approximations remains similarly small due to decrease of the linear term in (2.15). One concludes therefore that Eddington's approximation (2.15) remains valid to the accuracy of several per cent for all values of τ , $0 \leq \tau < \infty$. The derivative of $f(\tau)$ over τ , which is determined by the linear term in (2.15), contains a maximum inaccuracy of 15%, which takes place at $\tau < 0.3$.

3 Height of convection in an atmosphere with non-condensable greenhouse substances

In the previous section we showed that in the state of radiative equilibrium (absence of non-radiative energy fluxes) the dependence between the upward fluxes of thermal radiation into space, $F_\varepsilon \equiv F^+(\tau = 0)$, and at optical depth τ in the atmosphere, $F^+(\tau)$, in the spectral interval of resonance absorption of thermal radiation by atmospheric greenhouse substances conforms accurately to Eddington's approximation (2.15):

$$F^+(\tau) = F_e(1+k\tau), \quad k = \frac{3}{4}; \quad F_s^+(\tau) = F_e(1+k\tau_s). \quad (3.1)$$

Optical depth τ is unambiguously related to height z by Eq. (1.1). Brightness temperature T_b at height z in a given spectral interval $\Delta\lambda$ by definition is obtained from the following equality:

$$F^+(\tau) = \int_{\lambda_1}^{\lambda_2} I_p(\lambda, T_b) d\lambda \equiv \delta_{\Delta\lambda} \sigma T_b^4, \quad F_s^+ = \delta_{\Delta\lambda} \sigma T_s^4; \quad (3.2)$$

$$\int_0^{\infty} I_p(\lambda, T_b) d\lambda \equiv \sigma T_b^4, \quad \sum_{\Delta\lambda} \delta_{\Delta\lambda} = 1, \quad (3.3)$$

where σ is the Stephen-Boltzmann constant, $I_p(\lambda, T_b)$ is Planck's distribution, $\delta_{\Delta\lambda}$ is the relative share of thermal radiation within spectral interval $\Delta\lambda$ in the total blackbody radiation of temperature T_b . We assume $\delta_{\Delta\lambda}$ to be independent of temperature, neglecting the slow logarithmic broadening of absorption intervals with increasing temperature and taking into account the small relative change of temperature in the zone of convection of the modern atmosphere of Earth. As we show below, see (3.18), the relative change of temperature within the zone of convection remains small at large values of atmospheric optical thickness as well.

Brightness temperature T_b (3.2) characterises thermal radiation within a local volume. At large values of optical depth, $\tau \geq 1$, the local flux of thermal radiation into the upper hemisphere F^+ is formed due to emission of thermal photons by molecules within the same local volume. In such a case the population numbers n of the excited absorption bands of the greenhouse substances are related to brightness temperature T_b as $n = n_0 \exp(-\varepsilon/RT_b)$, where ε is the molar energy of the excited levels of the absorption bands, n_0 is concentration of the corresponding greenhouse substances. At $\tau < 1$ the upward flux of thermal radiation F^+ is formed in the local volume corresponding to $\tau \geq 1$ and significantly exceeds the local flux of radiation emitted by greenhouse substances. Thus the real temperature characterising population numbers n appears to be lower than the brightness temperature defined as in (3.2). However, for the modern atmosphere this inaccuracy can be safely neglected in the consideration of convective processes, as far as the zone of convection in the modern atmosphere correspond to $\tau \geq 1$, see below.

At any height local air temperature T is higher than brightness temperature T_b . The drop of air temperature observed in the troposphere means that there exist upward non-radiative fluxes of energy, not necessarily convective. The energy of such fluxes must be ultimately converted into energy of thermal radiation. If this is not the case, i.e. if the energy of non-radiative fluxes dissipate simply into the energy of thermal motion of air molecules, thus heating the atmosphere, the vertical temperature gradient ultimately disappears. It is the continuous pumping of thermal energy of molecular motion into thermal radiation, which allows the dissipated heat to leave into space and thus prevents air tem-

perature from becoming uniform at all heights. As far as energy may not be transmitted from a cooler object to the hotter one, such a pumping is only possible if the local air temperature T is higher than the brightness temperature T_b characterising population numbers of the excited energy levels of the absorption bands of the greenhouse substances: $T > T_b$. Brightness temperature may not be exactly equal to air temperature. Exact equality among the two temperatures corresponds to thermodynamic equilibrium between matter and radiation (as, for example, takes place in an enclosure with a small hole), when there is no net conversion from one type of energy (thermal motion of molecules) to another (thermal radiation) and, consequently, no temperature gradient. However, the difference between brightness temperature and air temperature may be relatively small as compared to some other thermal characteristics of the problem. Such a situation corresponds to approximate local thermodynamic equilibrium.

Using (3.1) and (3.2) we obtain the following expression of brightness temperature $T_b = T_b(z)$ in terms of optical depth $\tau = \tau(z)$:

$$T_b = T_s \left(\frac{1 + k\tau}{1 + k\tau_s} \right)^{1/4}. \quad (3.4)$$

As is clear from (3.4), brightness temperature is at its maximum at the Earth's surface, where it coincides with T_s , and decreases monotonously with growing height z due to monotonous decrease of $\tau(z)$ (1.1). The lapse rate G_b of brightness temperature (absolute magnitude of the negative temperature gradient) is maximum at the surface and decreases monotonously down to zero with growing z (diminishing τ).

In the course of inelastic collisions between air molecules and molecules of greenhouse substances, the kinetic energy of thermal motion of molecules is partially transmitted to excited energy levels of absorption bands of the greenhouse substances. This results in a drop of air temperature and appearance of a negative vertical gradient of air temperature, which necessarily brings about non-radiative fluxes of energy. Collisional interaction not only diminish the air temperature, but also increase brightness temperature, which, however, remains lower than air temperature at all heights, as discussed above. Thus, (3.4) can be viewed as giving the lower limit of brightness temperature at $z > 0$. As far as brightness and air temperatures coincide at the surface, $T_s = T_{bs}$, while at $z > 0$ we have $T > T_b$, the lapse rate of air temperature does not exceed that of brightness temperature, $G \leq G_b$, at any height $z \geq 0$.

When atmospheric concentrations of greenhouse substances and, consequently, optical thickness of the atmosphere τ_s , are low, the vertical non-radiative flux of energy is dictated by molecular diffusion and heat conductivity of air. As far as heat conductivity of air is very low, this flux is low as well, so that lapse rates of brightness temperature and air temperature at the Earth's surface practically coincide, $G_s \approx G_{bs}$.

When concentrations of greenhouse substances and optical thickness of the atmosphere increase, the lapse rates of brightness temperature and air temperature at the surface, $G_s \approx G_{bs}$, increase as well. As soon as the lapse rate of air temperature reaches

some critical threshold value Γ , which is dictated by properties of air and gravitational field of the planet, the atmosphere becomes convectively unstable and vertical convective processes set in. The non-radiative flux transporting latent and sensible energy rises then abruptly by 10^4 - 10^5 times as compared to the non-radiative flux maintained by air conductivity in the absence of convection. This abrupt change corresponds to transition from molecular to eddy diffusion processes in the atmosphere [6,11]. Convection is switched on at the Earth's surface when the optical thickness of the atmosphere τ_s , which plays the role of the so-called parameter of order of the considered problem, reaches a certain critical value $\tau_{s,\min}$, when $G_s = \Gamma$. With increasing $\tau_s > \tau_{s,\min}$ the zone of convection, where $G_b > \Gamma$, spreads upwards till yet larger and larger heights $z = z_c$, that are obtained from the equation $G_b(z_c) \geq G(z_c) = \Gamma$.

The gaseous atmosphere in the gravitational field of Earth approximately conforms to the condition of hydrostatic equilibrium and the equation of state for atmospheric air:

$$-\frac{dp}{dz} = MNg, \quad p = NRT, \quad \frac{dp}{dz} = -\frac{p}{h}, \quad h \equiv \frac{RT}{Mg}, \quad (3.5)$$

where p , N and T are pressure, molar concentration and temperature of air at height z , $M = 29 \text{ g mole}^{-1}$ is air molar mass assumed to be independent of height z in accordance with observations, $R = 8.3 \text{ J mole}^{-1} \text{ K}^{-1}$ is gas constant, $g = 9.8 \text{ m s}^{-2}$ is acceleration of gravity. Using (3.5) and (1.1) we obtain:

$$\frac{\tau}{\tau_s} = \frac{p}{p_s}, \quad (3.6)$$

where low index s refers to values at the Earth's surface. Relation (3.6) is valid for all gases with constant mass fraction, in particular, for non-condensable greenhouse substances like CO_2 . Due to the observed constant molar mass of air, the ratio p/p_s in (3.6) is equal to the corresponding ratio of partial pressures of such gases. Condensable greenhouse substances are considered in the next section.

We now derive the following relations from (3.5) and (1.1):

$$l = \frac{h}{\tau}, \quad h \equiv \frac{RT}{Mg}; \quad l_s = \frac{h_s}{\tau_s}, \quad h_s \equiv \frac{RT_s}{Mg}, \quad (3.7)$$

where $l \equiv l(z)$.

Using (1.1) we calculate the lapse rate of brightness temperature from (3.4):

$$G_b \equiv -\frac{dT_b}{dz} = \frac{1}{4} \frac{k}{(1+k\tau)^{3/4} (1+k\tau_s)^{1/4}} \frac{T_s}{l}, \quad l \equiv l(z). \quad (3.8)$$

Substituting l (3.7) into (3.8) we obtain:

$$G_b \equiv G_{\max} \alpha_b, \quad G_{\max} = \frac{Mg}{4R} = \frac{T_s}{4h_s} = 8.56 \text{ K km}^{-1},$$

$$\alpha_b = \frac{k\tau}{(1+k\tau)^{3/4} (1+k\tau_s)^{1/4}} \cdot \frac{T_s}{T}. \quad (3.9)$$

In (3.9) we express lapse rate of brightness temperature G_b as a product of dimensionless variable α_b and dimensional constant G_{\max} . As far as $\tau \leq \tau_s$, see (1.1), and $T \leq T_s$, we have $\alpha_b \leq 1$. The fundamental value of G_{\max} is a convenient scale for representing any temperature lapse rates G_i in the form of dimensionless variables $\alpha_i \equiv G_i / G_{\max}$.

As follows from (3.4) and (3.9), brightness temperature T_b and its lapse rate G_b are maximum at the surface. As already noted, $G < G_b$ at all heights, so that G_{\max} represents a limiting value for both lapse rate of brightness temperature G_b and that of air temperature G .

For lapse rate of brightness temperature at the surface G_{bs} we have:

$$G_{bs} = G_{\max} \alpha_{bs}, \quad \alpha_{bs} = \frac{k\tau_s}{1+k\tau_s}. \quad (3.10)$$

As is clear from (3.10), G_{bs} reaches its maximum value of G_{\max} at $\tau_s \rightarrow \infty$. Schwarzschild [12] noted that G_{\max} (3.9) is smaller than the so-called dry air adiabatic lapse rate, which for Earth equals $\Gamma_a = 9.8 \text{ K km}^{-1}$. This is sometimes interpreted as a contradiction [11], if one considers Γ_a as the value determining the onset of convection. If so, according to (3.10) the atmosphere would have been stable to convection, which is not the case. However, convection in the terrestrial atmosphere exists at the observed lapse rate $\Gamma = 6.5 \text{ K km}^{-1}$, which is smaller than both Γ_a and G_{\max} , $\Gamma = G_{\max} \alpha$, $\alpha = 0.77$. This value of Γ can be viewed as the critical lapse rate for the advent of convection in the terrestrial atmosphere. The condition $\Gamma < G_{\max}$ implied by (3.10) is therefore fulfilled in the terrestrial atmosphere.

According to (3.10), lapse rate of brightness temperature at the surface G_{bs} reaches the value of the observed convective lapse rate, $G_{bs} = G_s = \Gamma$, at $k\tau_s = 3.4$. At $k = 3/4$ this corresponds to $\tau_s = \tau_{s\min} = 4.5$, Fig. 1. This value of τ_s is sufficiently large for Edlington's approximation (2.15) to be accurate, so that use of $k = 3/4$ is valid when determining $\tau_{s\min}$.

Convection exists in that part of the atmosphere where air temperature lapse rate G exceeds the critical value Γ , $G > \Gamma$ [6]. As discussed above, the lapse rate of air temperature never exceeds the lapse rate of brightness temperature, $G \leq G_b$. Therefore, the height of the zone of convection does not exceed height z_c , where the diminishing lapse rate of brightness temperature reaches the critical value Γ . The value of z_c can be found by solving equation $G_b(z_c) = \Gamma$ or $\alpha_b(z_c) = \alpha$, where $G_b(z)$ is determined by (3.9).

This statement is independent of the degree of similarity between T and T_b , i.e. is independent of the assumption of local thermodynamic equilibrium. Collisional interaction between air molecules and molecules of greenhouse substances diminishes the difference between brightness temperature and air temperature, as well as the difference between their vertical gradients. Within the zone of convection, $z \leq z_c$, the lapse rate of

air temperature G is confined between G_b and Γ , $G_b > G \geq \Gamma$. Eq. (3.9) defines the maximum lapse rate of brightness temperature G_b , which is observed in the case of radiative equilibrium, i.e. in the absence of collisional excitation of molecules of greenhouse substances. In the reality, collisional excitation of molecules of greenhouse substances diminishes the lapse rate of brightness temperature G_b , driving G_b and G closer to each other. Value of z_c obtained from solving equation $G_b(z_c) = \Gamma$ with use of (3.9) can be thus viewed as an upper limit of the real height of convective zone. The obtained estimate of z_c becomes more and more accurate with growing T_s due to the fact that the dissipative power of non-radiative energy fluxes and, consequently, collisional excitation of molecules of greenhouse substances, are both limited by the absorbed solar radiation, which remains constant, while the upward flux of thermal radiation that determines brightness temperature grows infinitely with increasing T_s .

Within the zone of convection $z \leq z_c$, air temperature is determined by the observed convective lapse rate, $G \approx \Gamma$, which is approximately constant:

$$T = T_s - \Gamma z = T_s \left(1 - \frac{\alpha z}{4 h_s} \right), \quad \Gamma = G_{\max} \alpha, \quad G_{\max} = \frac{T_s}{4 h_s}, \quad \alpha = 0.77. \quad (3.11)$$

At $z > z_c$ there is no convection, and Eq. (3.11) may not be used for description of air temperature. All formulae derived with use of (3.11) are valid for $z \leq z_c$ only.

With an account made for the observed drop of air temperature with height (3.11), the vertical distribution of air pressure in the atmosphere under condition of hydrostatic equilibrium (3.5) assumes the form

$$p = p_s \left(1 - \frac{\alpha z}{4 h_s} \right)^{4/\alpha}, \quad z \leq z_c. \quad (3.12)$$

Using (3.6) we have

$$\tau = \tau_s \left(1 - \frac{\alpha z}{4 h_s} \right)^{4/\alpha}; \quad \frac{\tau}{\tau_s} = \frac{p}{p_s} = \left(1 - \frac{\alpha z}{4 h_s} \right)^{4/\alpha}, \quad z \leq z_c. \quad (3.13)$$

Equating $\alpha_b(z)$ (3.9) and the observed value of α we obtain:

$$\frac{k\tau}{(1+k\tau)^{3/4}(1+k\tau_s)^{1/4}} \cdot \frac{T_s}{T} = \alpha. \quad (3.14)$$

Variables τ and T in (3.14) are related to height z by Eqs. (3.13) and (3.11), respectively. Eq. (3.14) determines the maximum height of convective zone $z = z_c$ at different values of atmospheric optical thickness τ_s , Fig. 1. The dependencies of brightness temperature $T_b(z)$ on height z at different values of atmospheric optical thickness τ_s (3.4), (3.13), as well as the dependence of air temperature $T(z)$ on height z (3.11) are shown in Fig. 2. As far as the observed convective lapse rate corresponds to $\alpha = 0.77$, which is close to the maximum value of $\alpha = 1$, brightness and air temperatures remain close to each other within the convective zone, $z \leq z_c$, Fig. 2.

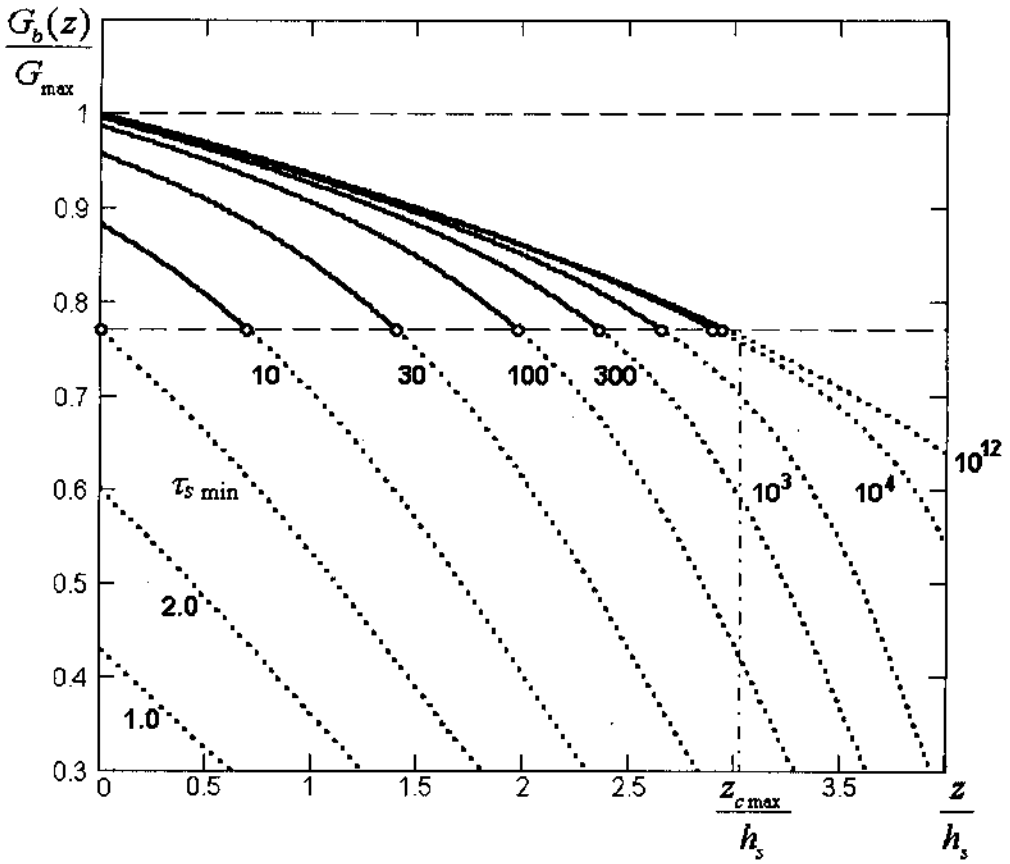


Figure 1: Lapse rate of brightness temperature G_b versus height z for different values of atmospheric optical thickness τ_s of non-condensable greenhouse substances, see (3.9), (3.11). The corresponding values of τ_s are shown below each curve; $h_s = RT_s/Mg$, see (3.7). The upper dashed line denotes maximum lapse rate $G_{\max} = Mg/4R$, the lower dashed line denotes the observed modern lapse rate of air temperature within the zone of convection, $\Gamma = 0.77G_{\max} = 6.5 \text{ K km}^{-1}$. The height z_c of the zone of convection for different values of τ_s is determined by the points of intersection of the curves $G_b(z)$ with the straight line $\Gamma = 0.77G_{\max}$, that are denoted by empty circles, see (3.14). The value of $z_{c\max} = 3.03h_s$, see (3.17), corresponds to the maximum height of convective zone observed at $\tau \gg 1$. As far as Eq. (3.11) is valid at $z \leq z_c$, parts of curves $G_b(z)$ at $z > z_c$ are denoted by dots. The onset of convection in the atmosphere occurs at $\tau_s = \tau_{s\min} = 4.5$, for which $z_c = 0$. Fig. 1 is invariant with respect to changes of surface temperature T_s .

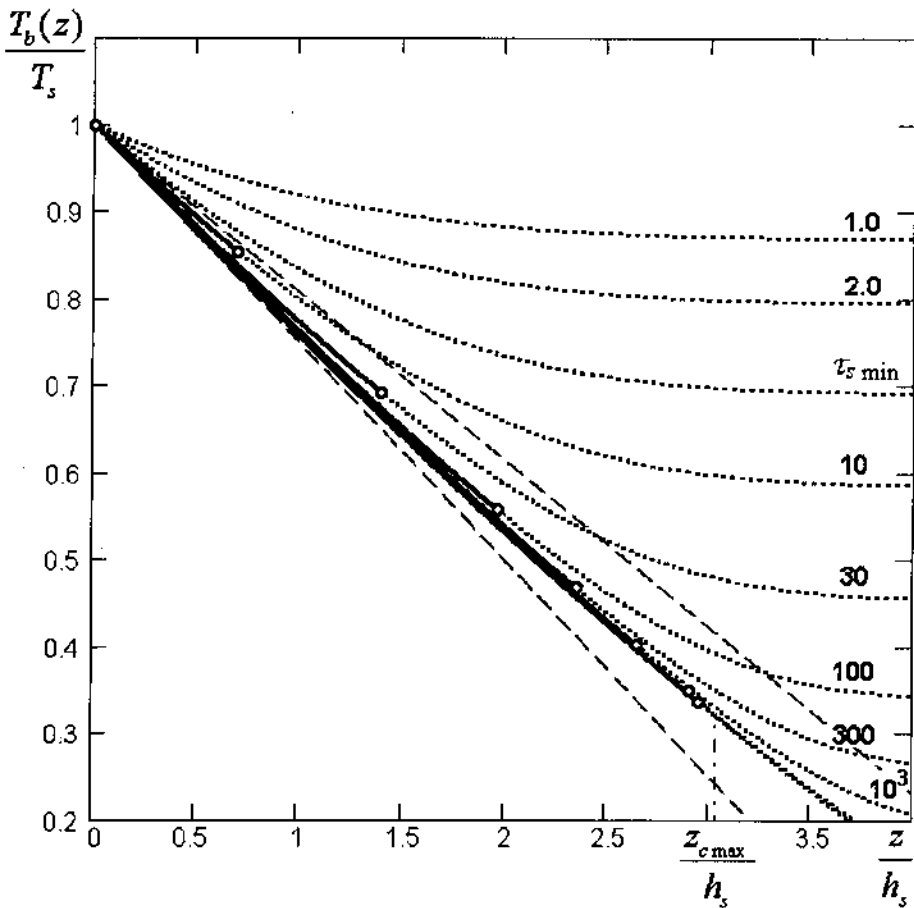


Figure 2: Dependence of brightness temperature $T_b(z)$ on height z , see (3.4), (3.13). T_s is the temperature of the Earth's surface; $h_s = RT_s / Mg$, see (3.7); the corresponding values of τ_s are shown above each curve except curves corresponding to $\tau_s = 10^4$ and $\tau_s = 10^{12}$. These curves practically coincide and go below the curve with $\tau_s = 10^3$. The upper dashed line denotes the observed lapse rate of air temperature for which $\alpha = 0.77$ ($\Gamma = 0.77G_{\max}$), see (3.11); the lower dashed line corresponds to the maximum lapse rate of brightness temperature G_{\max} , $\alpha = 1$. Empty circles correspond to the height of the convective zone z_c , at which the lapse rate of brightness temperature equates the convective lapse rate, see Fig. 1. As far as (3.13) is valid at $z \leq z_c$, parts of curves $T_b(z)$ at $z > z_c$ are denoted by dots, as in Fig. 1. Fig. 2 is invariant with respect to changes of surface temperature T_s .

At $\tau_s \gg 1$ Eq. (3.14) takes the following form:

$$\left(\frac{\tau}{\tau_s}\right)^{1/4} \frac{T_s}{T} = \alpha. \quad (3.15)$$

Assuming that convection may only slightly perturb the hydrostatic equilibrium of the atmosphere, one may use relations (3.5) to obtain from (3.15):

$$\left(\frac{p}{p_s}\right)^{1/4} \frac{T_s}{T} = \alpha. \quad (3.16)$$

The value of τ_s does not enter Eq. (3.16). This equation does not contain any information on the amount of greenhouse substances in the atmosphere. It only takes into account the approximate condition of hydrostatic equilibrium (3.5). Putting (3.11) and (3.12) into (3.16) we obtain the following expression for the maximum height of the zone of convection $z_{c \max}$ at $\tau_s \gg 1$:

$$z_{c \max} = 4h_s \frac{1}{\alpha} \left(1 - \alpha^{\frac{\alpha}{1-\alpha}}\right) = 3.03h_s = 25 \text{ km}. \quad (3.17)$$

The numeric value of 25 km corresponds to the modern Earth's surface temperature $T_s = 288 \text{ K}$, see (3.7). Eq. (3.17) describes the maximum height of the zone of convection in the terrestrial atmosphere with non-condensable greenhouse substances at the only absorbers of thermal radiation.

As far as the value of $\alpha = 0.77$ is close to unity, when obtaining $z_{c \max}$ from Eq. (3.16) one cannot neglect the dependence of temperature T on height z (3.11) as compared to dependence of air pressure p on height z . If one considers the atmosphere as approximately isothermal and assumes $T = T_s$ and $p = p_s \exp(-z/h_s)$ (see, e.g., [11]), the solution of (3.16) becomes $z_{c \max} = 4h_s \ln \frac{1}{\alpha} = 1.05h_s = 8.8 \text{ km}$, which is three times lower than the correct value (3.17).

Minimum values of air temperature $T_c \equiv T(z_c)$ and brightness temperature $T_{bc} \equiv T_b(z_c)$ at the top of the zone of convection, $z_c = z_{c \max}$ at large values of atmospheric optical thickness, $\tau_s \gg 1$, are obtained by substituting (3.17) into (3.11), and (3.17) into (3.13) and (3.4), respectively:

$$T_c = T_s \alpha^{\frac{\alpha}{1-\alpha}}, \quad T_{bc} = T_s \alpha^{\frac{1}{1-\alpha}} = \alpha T_c; \quad \frac{T_c}{T_s} = 0.42, \quad \frac{T_{bc}}{T_s} = 0.32, \quad \alpha = 0.77. \quad (3.18)$$

Thus, within the zone of convection both air temperature and brightness temperature retain their order of magnitude at any values of surface temperature T_s and optical thickness τ_s . The maximum relative difference between brightness and air temperature reaches $(1-\alpha) \times 100\% \approx 20\%$ at $\tau_s > 100$.

As is clear from (3.18), brightness temperature T_{bc} at the top of the convective zone grows proportionally to surface temperature T_s . Upward flux F_c^+ of thermal radiation at the top of the convective zone increases therefore proportionally to T_s^4 . The dissipative power of non-radiative convective fluxes of energy and, consequently, collisional excitation of the absorption bands of greenhouse substances, are limited by the incoming flux of solar radiation F_{in} , which does not change with increasing τ_s and T_s . (We note that in the non-stationary case the absorbed flux of solar radiation F_{in} is not equal to the outgoing flux of thermal radiation F_c , $F_{in} \neq F_c$, and also that $F_s^+ / F_{in} \neq 1 + k\tau_s$.) When the upward flux of thermal radiation at the surface is much larger than the absorbed flux of solar radiation, $F_s^+ = \sigma T_s^4 \gg F_{in}$, convective processes are unable to significantly perturb upward fluxes of thermal radiation and brightness temperature within the convective zone as compared to the case of radiative equilibrium. With growing τ_s and T_s the radiative equilibrium spreads from the stratosphere to the entire atmosphere, so that Eddington's approximation becomes valid for the entire atmosphere as well. Therefore, Eqs. (3.17) and (3.18) obtained with use of Eddington's approximation yield at $F_s^+ \gg F_{in}$ and $\tau_s \gg 1$ exact values of z_c , T_{bc} and T_c rather than an upper estimate of z_c and lower estimates of T_{bc} and T_c .

On Venus the ratio of the upward flux of thermal radiation at the surface F_s^+ to the outgoing flux of thermal radiation into space $F_e = F_{in}$ equals $F_s^+ / F_e = 98$ [3]. In accordance with Eddington's approximation (3.1) this corresponds to $\tau_s = (4/3)(F_s^+ / F_e) = 131 \gg 1$. Thus, Eqs. (3.17) and (3.18) can be applied to Venus with a high accuracy. Taking for Venus $h_s = 15.6$ km, $G_{max} = 11.4$ K km⁻¹ ($T_s = 730$ K, $M_{CO_2} = 44$ g mole⁻¹, $g = 8.6$ m s⁻² [3,13,14]), the observed air temperature lapse rate $\Gamma \approx 8$ K km⁻¹ [2], we obtain for Venus $\alpha \equiv \Gamma / G_{max} = 0.70$. Putting these values into (3.17), we obtain the following value for the height $z_{c,max}$ of the convective zone on Venus, $z_{c,max} = 50.4$ km. This value agrees very well with the observed height of cloudiness on Venus, which equals (52÷53) km. From (3.18) we calculate air temperature at $z_{c,max}$ to be $T_c = 318$ K (44°C).

We note that the height of convection z_c is generally less than the height of the tropopause z_{tp} , $z_c < z_{tp}$. The latter is usually defined as the point where the lapse rate of air temperature approaches zero. As is clear from Fig. 2, the lapse rate of brightness temperature and, consequently, lapse rate of air temperature may remain substantial for $z > z_c$, being nevertheless smaller than the critical value Γ responsible for the presence of convection. This effect is especially pronounced for small values of $\tau_s \sim \tau_{s,min}$, see Fig. 2.

4 Height of convection in an atmosphere with condensable greenhouse substances

The dependence of saturated water vapour pressure p_L on temperature is described by Clausius-Clapeyron equation (everywhere below the low index L denotes variables corresponding to water vapour):

$$p_L = p_{L_s} \exp\left\{\frac{T_L}{T_s} - \frac{T_L}{T}\right\}, \quad T_L \equiv \frac{Q}{R} \approx 5300 \text{ K}, \quad (4.1)$$

where $Q \approx 44 \text{ KJ mole}^{-1}$ is the latent heat of evaporation of one mole of water vapour. The value of Q changes only insignificantly with temperature, decreasing by 10% when temperature rises by 100°C . The value of Q used in (4.1) corresponds to the observed global mean surface temperature $+15^\circ\text{C}$. Everywhere below it is considered to be approximately independent of temperature T .

In order to consider changes of the outgoing thermal radiation with temporal changes in the surface temperature, it is convenient to denote as T_{s0} and p_{s0} the initial values of surface temperature and pressure that correspond to the initial state of the existing thermal equilibrium of the planet. Then T and p will be corresponding values of temperature and pressure at arbitrary moments of time t and height z , while T_s and p_s will be corresponding values of temperature and pressure at the surface at an arbitrary moment of time t :

$$\begin{aligned} p &\equiv p(z, t), & T &\equiv T(z, t); \\ p_s &\equiv p(0, t), & T_s &\equiv T(0, t); \\ p_{s0} &\equiv p(0, 0), & T_{s0} &\equiv T(0, 0). \end{aligned} \quad (4.2)$$

Generally, we will apply low index s to refer, as above, to Earth's surface and low index 0 to refer to values corresponding to initial state of thermal equilibrium.

Relation (4.1) can be viewed as determining partial pressure of saturated water vapour at an arbitrary height z at an arbitrary moment of time t . It is easy to see that (4.1) conforms to the following invariance rule:

$$p_s \exp\left(\frac{T_L}{T_s}\right) = p_{s0} \exp\left(\frac{T_L}{T_{s0}}\right). \quad (4.3)$$

If we demand that water vapour in the atmosphere is saturated and is in hydrostatic equilibrium, see (3.5), the corresponding lapse rate Γ_L of air temperature at any altitude z is obtained from (4.1):

$$\Gamma_L \equiv -\frac{dT}{dz} = \frac{T}{H_L}. \quad (4.4)$$

Solution of (4.4) is

$$T = T_s e^{-\frac{z}{H_L}}, \quad H_L \equiv \frac{RT_L}{gM} = 154 \text{ km}.$$

Due to the large value of temperature T_L (4.1), for a relatively wide interval of surface temperature changes Γ_L is practically constant (independent of z) in the entire troposphere:

$$\Gamma_L = \frac{T_s}{h_L} = \alpha_L G_{\max}, \quad \alpha_L \equiv \frac{4T_s}{T_L}, \quad \alpha_{L0} \equiv \frac{4T_{s0}}{T_L} = 0.22; \quad \Gamma_{L0} = 1.9 \text{ K km}^{-1}. \quad (4.5)$$

As can be seen from (4.5), the lapse rate of air temperature in an equilibrium atmosphere saturated with respect to water vapour is determined by the same dimensional combination as the maximum lapse rate G_{\max} , see (3.9), (3.11). The appearance of a new temperature scale T_L uniquely determines the magnitude of the dimensionless lapse rate α_L , the modern value of which appears to be approximately three times lower than the observed value of α , see (3.11).

If the lapse rate of air temperature $G = \alpha G_{\max}$ were smaller than $\Gamma_L = \alpha_L G_{\max}$, $\alpha < \alpha_L$, atmospheric water vapour would be unsaturated at all heights z . In such a case there would be no vertical fluxes of matter in the atmosphere, that is, no continuous upward fluxes of evaporating water vapour and no compensating downward fluxes of precipitated water in liquid and water phases. At present $G \approx \Gamma = 3.5\Gamma_L$ and $\alpha = 3.5\alpha_{L0}$. This means the atmospheric water vapour is on average saturated, but is not in hydrostatic equilibrium, i.e. condition (3.5) for water vapour is not fulfilled. The vertical change of water vapour pressure is not compensated by the corresponding change in the weight of water vapour in the atmospheric column. Such a situation can be only maintained if there exist upward fluxes of water vapour. These fluxes form in the course of continuous evaporation of water vapour from the hydrosphere and are compensated by downward fluxes of precipitation that form when the water vapour condenses in the upper cooler layers of the atmosphere.

As can be seen from (4.5), the saturated lapse rate α_L grows proportionally to the surface temperature T_s , while lapse rate α of air temperature remains constant. Thus, with growing surface temperature T_s , the value of α_L approaches α , $\alpha_L \rightarrow \alpha$. In the result, atmospheric water vapour becomes unsaturated, evaporation and precipitation fluxes weaken and ultimately cease completely when $\alpha_L \rightarrow \alpha$.

As far as the observed lapse rate $G \approx \Gamma$ of air temperature is steeper than Γ_L , $\Gamma > \Gamma_L$, distribution of water vapour in the atmosphere differs from the hydrostatic equilibrium distribution (3.5). Calculating dp_L/dz with use of (4.1), we find:

$$\frac{dp_L}{dz} = -\frac{p_L}{h} \beta, \quad \beta = \alpha \frac{T_L}{4T} \approx \frac{\alpha}{\alpha_L}. \quad (4.6)$$

Using (3.5), the dependence of T on z (3.11), (3.12) and (4.6) we obtain the following approximate dependence of water vapour partial pressure p_L on height z :

$$p_L = p_{Ls} \left(1 - \alpha_L \frac{z}{4h_{Ls}} \right)^{\frac{4}{\alpha_L}}, \quad (4.7)$$

where

$$\alpha_L \equiv \frac{4T_s}{T_L}, \quad h_{L_s} \equiv \frac{h_s}{\beta}. \quad (4.8)$$

Note that (4.7) is formally identical to (3.12) if one substitutes α for α_L and h_s for h_{L_s} . Comparing (4.7) and (3.12) we observe that

$$\frac{p_L}{p_{L_s}} = \left(\frac{p}{p_s} \right)^\beta, \quad (4.9)$$

where p_s and p (3.12) are air pressure at the surface and at an arbitrary height z , respectively.

As is clear from (4.9), the modern distribution of water vapour in the atmosphere is compressed by $\beta_0 = \alpha/\alpha_{L0} = 3.5$ times as compared to distribution of air [11]. The scale height of water vapour distribution in the modern atmosphere equals $h_{L0} = h_{s0}/\beta_0 = 2.4$ km, which agrees well with observations [2,11,15]. Compression of water vapour in the modern atmosphere significantly violates hydrostatic equilibrium and brings about intensive dynamic fluxes of hydrosphere evaporation, upward transport of evaporated water and compensating downward fluxes of precipitation.

Using definition (1.1) for τ_L , Eq. (4.7) for p_L and equation of state $p_L = N_L RT$ (3.5), we obtain the following relations:

$$\frac{\tau_L}{\tau_{L_s}} = \frac{p_L}{p_{L_s}}, \quad z \leq z_{L_c}; \quad (4.10)$$

$$l_L = \frac{h_L}{\tau_L}, \quad h_L \equiv \frac{h}{\beta}, \quad h \equiv \frac{RT}{Mg}; \quad l_{L_s} = \frac{h_{L_s}}{\tau_{L_s}}, \quad h_{L_s} \equiv \frac{h_s}{\beta}, \quad h_s \equiv \frac{RT_s}{Mg}. \quad (4.11)$$

Relations (4.6)-(4.11), where β is approximately put equal to $\alpha/\alpha_{L0} = 3.5$, is valid to the accuracy of about 6% due to abrupt exponential drop of pressure with z caused by the large value of β . Relations (4.10) and (4.11) for condensable greenhouse substances appear to be identical to the corresponding relations (3.6) and (3.7) for non-condensable greenhouse substances if one substitutes h and h_s (3.7) by h_L and h_{L_s} (4.9), respectively, the latter being β times lower than the former. Similarly, the maximum value of brightness temperature lapse rate $G_{\max} = T_s/4h_s$ (3.9) becomes β times larger, $G_{L\max} = T_s/4h_{L_s} = G_{\max}\beta = 30 \text{ K km}^{-1}$.

The expression for maximum height of the zone of convection determined by water vapour, $z_{L_c\max}$, is obtained from (3.14) by a similar substitution $\alpha \rightarrow \alpha_L$, $\tau \rightarrow \tau_L$ and $\tau_s \rightarrow \tau_{L_s}$, where $\alpha_L = 0.22$ (4.5), τ_L is determined by (4.10) and (4.7) and temperature T is determined by (3.11). The onset of convection in the terrestrial atmosphere due to the greenhouse effect of water vapour occurs at $\tau_{L_s\min} = 0.38$, Fig. 3, which is smaller than $\tau_{s\min} = 4.5$ for non-condensable greenhouse substances like CO_2 , Fig. 1. The idea that due to the above discussed compression of water vapour in the modern atmosphere

water vapour is more efficient with respect to switching on convection than non-condensable absorbers like CO_2 , was put forward in [11].

Note that in the intervals of $\tau_{L_s} < \tau_{L_s \text{ min}}$ corresponding to $\Gamma > G > \Gamma_L$ water vapour is already not in hydrostatic equilibrium (i.e. there are fluxes of evaporation and precipitation), while atmospheric convection is still absent.

At large values of optical thickness $\tau_s \gg 1$ Eqs. (3.17) and (3.18) retain their form for water vapour if one substitutes $h_s \rightarrow h_{L_s} = h_s / \beta$ and $\alpha \rightarrow \alpha_L = \alpha / \beta$:

$$z_{L_c \text{ max}} = 4h_{L_s} \frac{1}{\alpha_L} \left(1 - \alpha_L \frac{\alpha_L}{1 - \alpha_L} \right) \approx 4 \frac{h_s}{\beta} \ln \frac{\beta}{\alpha} = 6.3 \frac{h_s}{\beta}. \quad (4.12)$$

For the modern observed value of $\beta_0 = 3.5$ we have $z_{L_c \text{ max}} = 15 \text{ km}$.

Note that Eq. (3.17), as well as Figures 1 and 2, are invariant with respect to changes in surface temperature T_s in that at any value of T_s the result $z_{c \text{ max}} / h_s = 3.03$ (3.17) is true. Eq. (4.12) and Fig. 3 are not invariant with respect to T_s , as far as the compression factor β decreases proportionally to T_s , see (4.8). Fig. 3 corresponds to surface temperature $T_s = 288 \text{ K}$. The value of β reaches unity, $\beta = 1$, when surface temperature increases 3.5-fold as compared to its modern value. At this point atmospheric water vapour becomes unsaturated within the entire convective zone and formulas (3.17) and (4.12) coincide.

For air temperature at $z = z_{L_c \text{ max}}$ we have:

$$T_{L_c} = T_s \alpha_L \frac{\alpha_L}{1 - \alpha_L} = T_s \times 0.65, \quad (4.13)$$

i.e. at $T_s = 288 \text{ K}$ we have $T_{L_c} = 188 \text{ K} = -85^\circ \text{C}$.

In the terrestrial atmosphere atmospheric content of water vapour is strongly dependent on surface temperature [16] rising approximately twofold per each ten degrees of surface temperature increase, see (4.1). The maximum values of atmospheric optical thickness of water vapour are therefore to be observed in the equatorial regions. In accordance with (4.12) and (4.13), in these regions one also observes minimum air temperatures of the order of -80°C and, consequently, maximum height of the zone of convection of the order of 15-16 km.

Optical thickness of atmospheric water vapour τ_{L_s} , as well as partial pressure of water vapour at the Earth's surface p_{L_s} , see (4.10), increase exponentially with growing increment of surface temperature $\theta \equiv T_s - T_{s0}$, see (4.3):

$$p_{L_s} = p_{L_s0} \exp\left(\frac{\theta}{\theta_s}\right), \quad \theta \equiv T_s - T_{s0}, \quad \theta_s \equiv \alpha_{L0} \frac{T_s}{4}, \quad \theta_{s0} = \alpha_{L0} \frac{T_{s0}}{4} = 18 \text{ K}. \quad (4.14)$$

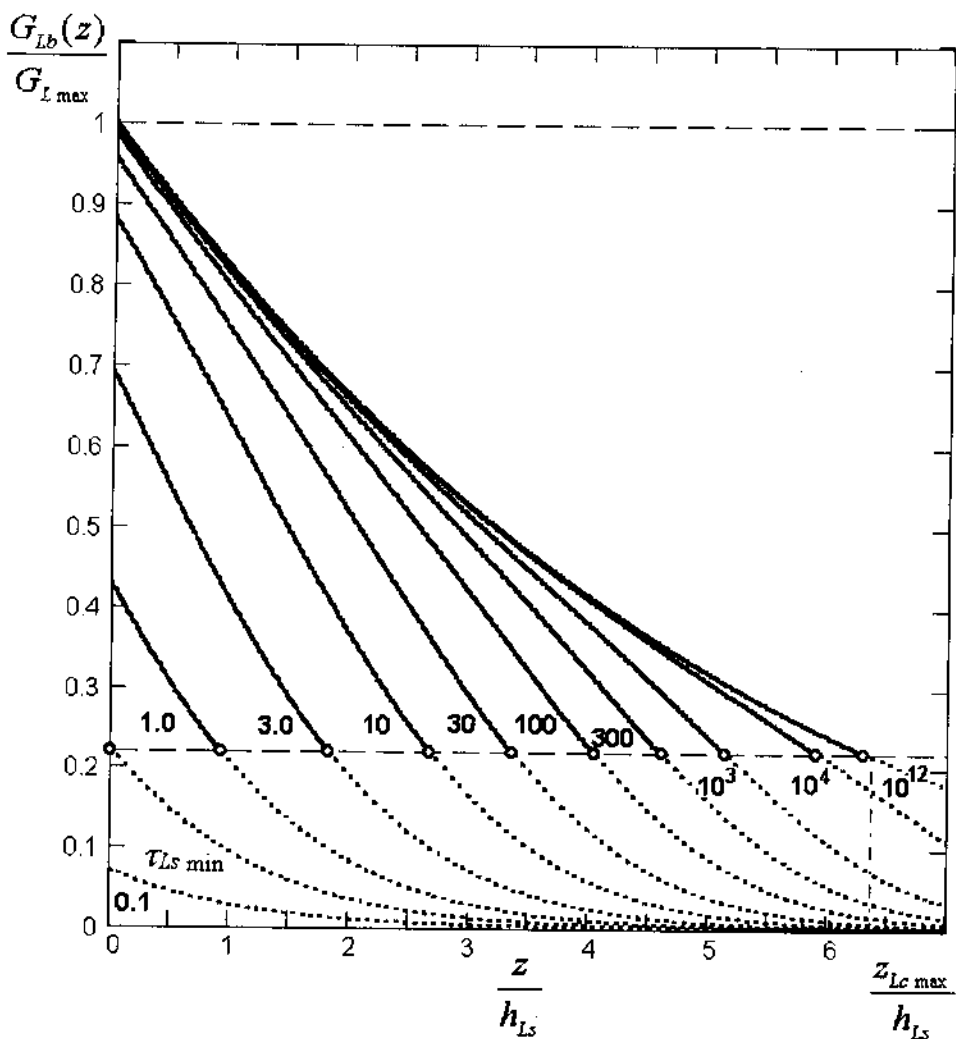


Figure 3: Lapse rate of brightness temperature G_{lb} versus height z for different values of atmospheric optical thickness τ_{L_s} of water vapour, see (4.12). The corresponding values of τ_{L_s} are shown below each curve; scale height $h_{L_s} = 2.4$ km. The upper dashed line denotes maximum lapse rate $G_{L_{\max}} = G_{\max} \beta = 30 \text{ K km}^{-1}$, $\beta = 3.5$; the lower dashed line denotes the observed modern lapse rate of air temperature within the zone of convection, $\Gamma = \alpha_L G_{L_{\max}} = 6.5 \text{ K km}^{-1}$, $\alpha_L = 0.22$, see (4.5). The height z_{L_c} of the zone of convection for different values of τ_{L_s} is determined by the points of intersection of the curves $G_{L_b}(z)$ with the straight line $\Gamma = \alpha_L G_{L_{\max}}$ (denoted as empty circles). The value of $z_{L_c \max} = 6.3 h_{L_s}$, see (4.12), corresponds to the maximum height of convective zone observed at $\tau_{L_s} \gg 1$ and $T_s = 288$ K. The onset of convection in the atmosphere occurs at $\tau_{L_s} = \tau_{L_s \min} = 0.38$, for which $z_{L_c} = 0$

This means that at $\tau_L \gg 1$, in accordance with the results obtained in the previous sections, see (2.15) and (3.1), the ratio of the outgoing flux of thermal radiation F_{L_e} to the upward flux of thermal radiation at the surface $F_{L_e}^+$ within the absorption interval of water vapour drops exponentially with growing θ :

$$\frac{F_{L_e}}{F_{L_e}^+} \propto \frac{1}{\tau_L} = \frac{1}{\tau_{L_0}} \exp\left(-\frac{\theta}{\theta_s}\right), \quad \theta_s = (18 \text{ K}) \cdot \frac{T_s}{T_{s0}}. \quad (4.15)$$

The ratio of the effective brightness temperature corresponding to radiation emitted by water vapour into space, T_{L_e} , to the surface temperature T_s , also decreases exponentially, although with a smaller exponent:

$$\frac{T_{L_e}}{T_s} \propto \frac{1}{(\tau_{L_0})^{1/4}} \exp\left(-\frac{\theta}{4\theta_s}\right). \quad (4.16)$$

Air temperature at the upper radiative layer of the atmosphere corresponding to $\tau_L = 1$, $T_1 \equiv T(\tau_L = 1)$, can be obtained from (4.10), (4.3) and (4.1):

$$\frac{\tau_{L_e}}{p_{L_e}} p_L = 1, \quad \frac{\tau_{L_0}}{p_{L_0}} = \frac{\tau_{L_e}}{p_{L_e}}, \quad T_1 = \frac{T_s}{1 + \frac{T_s}{T_L} \ln \tau_{L_e}} = \frac{T_{s0}}{1 + \frac{T_{s0}}{T_L} \ln \tau_{L_0}}. \quad (4.17)$$

As is clear from (4.17), T_1 is independent of either T_s or θ . This result is physically transparent. Condition $\tau_L = 1$ unambiguously determines concentration and partial pressure of the greenhouse substance, see (4.10), while partial pressure of saturated water vapour is uniquely related to temperature in accordance with Clausius-Clapeyron equation (4.1). Thus, even if one assumes that initially there exists local thermodynamic equilibrium in the vicinity of $\tau_L = 1$, i.e. that $T_1 \approx T_{L_0}$ for $T_s = 288 \text{ K}$, as soon as surface temperature increases by $\theta = 150 \text{ K}$, i.e. by 1.5 times, the difference between T_1 and T_{L_e} , in accordance with (4.17), will reach nearly one order of magnitude.

Moreover, the magnitude of change in the outgoing radiation flux, which is of primary interest, will be determined namely by the difference between T_1 and T_{L_e} , i.e. by the degree to which the local thermodynamic equilibrium is broken. It follows that even at very small temperature changes the assumption of local thermodynamic equilibrium together with the assumption that water vapour is saturated at all heights may not be used for analysing changes in the outgoing long-wave radiation, as it will invariably yield a constant outgoing long-wave radiation flux [2, 7, 14, 17–19].

5 Conclusions

In this paper we have shown that the height of the zone of convection in the atmosphere with condensable and non-condensable absorbers is independent of optical thickness of the atmosphere τ_s at large values of the latter. As a consequence, the accuracy of Ed-

dington's approximation for description of the ratio between outgoing flux of thermal radiation F_e and the upward flux of thermal radiation at the surface F_s^* increases with growing optical thickness of the atmosphere, irrespective of what non-radiative heat fluxes are present in the lower troposphere, including convection. This conclusion is based on the fact that the energy of non-radiative convective heat fluxes that create distortion from radiative equilibrium in the lower atmosphere is limited by the value of the absorbed solar radiation. On the contrary, upward radiative flux at the surface may increase without bounds with growing surface temperature. Thus, with growing τ_s radiative processes in the atmosphere become more and more powerful, while non-radiative processes remain limited by the absorbed solar flux. As a result, radiative equilibrium becomes more and more exact.

Here we summarise briefly the physics of the results obtained. According to Eddington's approximation (3.1), brightness temperature T_b (3.4) at $\tau \gg 1$ becomes a function of ratio τ/τ_s : $T_b = T_s \varphi(\tau/\tau_s)$. The condition of hydrostatic equilibrium $\tau/\tau_s = p/p_s = \Phi(z/h_s)$ (3.5) means that ratio τ/τ_s is a function of ratio z/h_s . (This statement is valid for the condensable greenhouse substances as well due to the practically constant value of the compression factor β , see Section 4). We thus obtain that $T_b = T_s f(z/h_s)$ is independent of atmospheric optical thickness τ_s . The lapse rate of brightness temperature $G_b = (T_s/h_s) f'(z/h_s) = G_{\max} \alpha_b(z/h_s)$, where $G_{\max} = T_s/4h_s$ and $\alpha_b(z/h_s) \equiv 4f'(z/h_s)$, is also a function of ratio z/h_s and independent of τ_s . Equating the lapse rate of brightness temperature G_b to the critical convective lapse Γ , which is a constant multiple of G_{\max} , $\Gamma = \alpha G_{\max}$, we obtain height z_c of the convective zone from the equation $\alpha_b(z_c/h_s) = \alpha$. This gives $z_c = h_s \alpha_b^{-1}(\alpha)$, where $\alpha_b^{-1}(x)$ is the reverse function of $\alpha_b(x)$. We conclude that at $\tau_s \gg 1$ the height of convective zone z_c is independent of τ_s either. Thus, independence of z_c , T_c and T_{bc} of τ_s at $\tau_s \gg 1$ is the consequence of Eddington's approximation and hydrostatic equilibrium of the gaseous atmosphere in the gravitational field of Earth.

Optical depth $\tau = \tau_c$ at the top of the convective zone corresponds to the optical thickness of the stratosphere. As far as the ratio $z_c/h_s = \alpha_b^{-1}(\alpha)$ is independent of τ_s , optical thickness of stratosphere τ_c grows proportionally to the optical thickness of the entire atmosphere τ_s , $\tau_c = \tau_s (p_c/p_s) = \tau_s \Phi(z_c/h_s)$. This means that there is no fixed optical depth that would be attainable by convective processes at any value of τ_s , see also [9], p. 304. That stratospheric optical thickness τ_c increases with growing τ_s means that contribution of the dissipative power of convective processes into formation of the outgoing flux of thermal radiation into space tends to zero at $\tau_s \rightarrow \infty$.

One could imagine a situation when a certain finite amount of energy of the non-radiative fluxes dissipates in the upper radiative layer of the atmosphere, e.g. at $\tau \sim 1$, irrespective of the magnitude of the atmospheric optical thickness τ_s . In such a case the

outgoing flux of thermal radiation into space F_e would not diminish with growing T_s , but reached a finite plateau determined by the value of non-radiative dissipative power at $\tau \sim 1$. In other words, convective processes would "capture" a finite part of solar radiation at the surface in the form of dynamic energy and transport it till the upper radiative layer of the atmosphere, from which it would dissipate directly into space. This "convective" part of solar energy might be then compared to planetary albedo, i.e. to the amount of solar energy reflected by the planet. The difference between albedo and convection dissipating at $\tau \sim 1$ is that the former leaves the planet as short-wave radiation, while the latter ultimately dissipates into thermal long-wave radiation. However, as well as the part of solar energy reflected by the planet, the dynamic energy of convective fluxes that dissipates at the top of the atmosphere leaves into space without performing any work on the planet's surface or in the atmosphere and without participating in formation of temperature regime at the planet's surface. As we have shown in this paper, such a situation is physically impossible.

On a planet with liquid hydrosphere the outgoing thermal radiation into space decreases exponentially with growing surface temperature. This is a consequence of an exponential growth with surface temperature of the optical thickness of atmospheric water vapour and cloudiness, which covers the entire thermal spectrum of terrestrial radiation emitted to space.

In the case of an infinite hydrosphere, the stationary state when the outgoing thermal radiative flux equals the flux of absorbed solar radiation is never attained by the atmosphere. If we start from the observed equilibrium conditions, with increasing surface temperature the outgoing long-wave radiation flux will be always less than the absorbed flux of solar radiation. The absorbed flux of solar radiation, always exceeding that of the outgoing long-wave radiation, will then be spent on generation of dynamic processes in the atmosphere, including flux of evaporation and latent heat transfer. Governed by radiative processes, the decrease in the outgoing thermal flux occurs instantaneously in reaction to increase in T_s . In the meantime, the characteristic times of dissipation of dynamic processes are finite. This allows one to assume that the rate of accumulation of cloudiness in the atmosphere will exceed the rate of precipitation. This will lead to increase in atmospheric optical thickness not only in the absorption interval corresponding to water vapour, but over entire thermal spectrum, as far as cloudiness absorbs radiation rather evenly over the entire thermal spectrum. Such a situation will correspond to a runaway greenhouse effect.

References

- [1] D. Michalas and B.W. Michalas, *Foundations of radiation hydrodynamics*, Oxford University Press, New York (1984).
- [2] R.M. Goody and Y. L.Yung, *Atmospheric radiation, the theoretical basis*. 2nd edn. Oxford University Press, New York, 1989, 519.
- [3] J. Mitchell, *Rev. Geophys.* **27** (1989) 115.
- [4] S.H. Schneider, *Science*. **243** (1989) 771.
- [5] S. Chandrasekhar, *An Introduction in the Study of Stellar Structure*, Dover, New York, 1957.
- [6] V. Ramanathan and J.A. Coakley, *Rev. Geophys. Space Phys.* **16** (1978) 465.
- [7] S. Nakajima *et al.*, *J. Atmos. Sci.* **49** (1992) 2256.
- [8] E.A. Milne, *Handbuch Astrophys.* **3** (1930) 65.
- [9] V.G. Gorshkov and A.M. Makarieva, *Atmos. Chem. Phys. Discuss.* **2** (2002) 289.
- [10] E. Hopf, *Mathematical problems of radiative equilibrium*, Cambridge Univ. Press, Cambridge, 1934.
- [11] C.P. Weaver and V. Ramanathan, *J. Geophys. Res.* **100D** (1995) 11585.
- [12] K. Schwarzschild, *Selected Papers on the Transfer of Radiation*. Ed. D.H. Menzel, Dover, New York, 1966, 25–34.
- [13] C.W. Allen, *Astrophysical Quantities*, Athlone Press, London, 1955.
- [14] J.F. Kasting, *Icarus*. **74** (1988) 472.
- [15] H.P. Pruppacher and J.D. Klett, *Microphysics of clouds and precipitation*, D Reidel Publishing Company, Dordrecht, 1978.
- [16] A. Raval and V. Ramanathan, *Nature*. **342** (1989) 758.
- [17] M. Komabayasi, *J. Meteorol. Soc. Japan*. **45** (1967) 137.
- [18] A.P. Ingersoll, *J. Atmos. Sci.* **26** (1969) 1191.
- [19] T. Pujol and G.R. North, *J. Atmos. Sci.* **59** (2002) 2801.

ФИЗИКА АТОМНОГО ЯДРА
И ЭЛЕМЕНТАРНЫХ ЧАСТИЦ

NUCLEAR AND PARTICLE
PHYSICS

МАТЕРИАЛЫ
XXXVI ЗИМНЕЙ ШКОЛЫ ПИЯФ,
25 ФЕВРАЛЯ – 3 МАРТА, 2002

PROCEEDINGS OF THE
XXXVI WINTER SCHOOL OF PNPI,
FEBRUARY 25 – MARCH 3, 2002

Ответственный за выпуск: *В. А. Гордеев*

Научные редакторы: *Б. Л. Бирбраир, В. И. Велижанин, В. А. Гордеев,
В. И. Исаков, В. А. Кудрявцев, И. В. Лопатин, И. А. Митропольский,
В. Ю. Петров, М. Г. Рыскин, Г. Е. Солякин, Ю. М. Шабельский*

Литературные редакторы: *Н. С. Морозова, Е. Ю. Оrobeц, И. А. Слободни*

Технический редактор: *Т. А. Парфеева*

Компьютерная обработка и верстка: *Г. В. Степанова,
Е. В. Веселовская, Т. А. Парфеева*

Печатается по решению Координационного совета ПИЯФ
от 12 января 2001 г.

Издательская лицензия ЛР №020735 от 25 декабря 1998 г.

Отпечатано в типографии ПИЯФ РАН

188300, Гатчина Ленинградской обл., Орлова роща
Зак 89, тир.350, уч.-изд. л.46; 25.12.2002 г.
Формат 70x100 1/16, печать офсетная