

# Quantum Anomalies

Quantum anomalies are among the most interesting and far reaching phenomena in QFT. The phenomenon is the following: Let the action (or Hamiltonian) possess, at the classical level, invariance under certain continuous transformations. It means that there is a corresponding conserved Nöther current,  $\partial_\mu J_\mu^{\text{Nöther}} = 0$ .

Now we switch in quantum fluctuations of the field. Fluctuations may have infinitely high Fourier components (= momenta), and one has to regularize the theory somehow. One tries to implement the regularization so as not to spoil the symmetries of the theory. For example, it is preferable not to violate the translational and rotational invariance (but it is violated by lattice regularization.) In gauge theories, invariance under gauge transformations is sacred (but it is violated by cutting off momenta at  $|k| > k_{\text{max}}$ ). These symmetries are preserved by dimensional regularization (but it is not applicable to chiral fermions). One of the best regularization is by proper time (but it is not universally applicable either).

Q.: What symmetry is broken by Pauli–Villars large-mass regularization?

Whatever the regularization, the hope is that when one takes the cutoff to  $\infty$ , symmetry is restored, and quantum corrections do not violate the conservation of the classical Nöther currents. However, there are cases when it is impossible, as a matter of principle, – not because of our limited wisdom. **If a symmetry is violated at the quantum level, it is called a quantum anomaly.**

Most anomalies occur in theories with fermions, although there are examples of purely bosonic anomalies. We shall work with relativistic fermions but in Euclidean space-time:

$$S_{\text{Dirac}} = \int d^d x \psi^\dagger (\gamma_\mu \nabla_\mu + im) \psi, \quad \nabla_\mu = \partial_\mu - iA_\mu,$$

where the Dirac matrices  $\gamma_\mu$ ,  $\mu = 1, \dots, d$ , obey the **Clifford** algebra

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\delta_{\mu\nu}, \quad \gamma_\mu^\dagger = \gamma_\mu.$$

In  $d = 1 + 1$  and  $d = 2 + 1$   $\gamma_\mu$ 's are  $2 \times 2$  matrices that can be chosen to be Pauli matrices (they satisfy the needed algebra):

$$d = 2 : \quad \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_2,$$

$$d = 3 : \quad \gamma_1 = \sigma_1, \quad \gamma_2 = \sigma_2, \quad \gamma_3 = \sigma_3.$$

In  $d = 3 + 1$   $\gamma_\mu$ 's are  $4 \times 4$  matrices which can be chosen as

$$d = 4 : \quad \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu^- \\ \sigma_\mu^+ & 0 \end{pmatrix}, \quad \sigma_\mu^\pm = (\pm i\sigma_i, \mathbf{1}_2).$$

In even number of dimensions a special role is played by a matrix which we shall generically denote by  $\gamma_5$ :

$$\left. \begin{array}{l} d = 2 : \quad \gamma_5'' = -i\gamma_1\gamma_2 \\ d = 4 : \quad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 \end{array} \right\} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_\mu\gamma_5 + \gamma_5\gamma_\mu = 0, \quad \gamma_5^\dagger = \gamma_5, \quad \gamma_5\gamma_5 = 1.$$

The glossary for passing from Minkowski to the Euclidean space is

$$\begin{aligned} t_E &= it_M, & A_{4E} &= -iA_{0M}, & \psi_E^\dagger &= i\bar{\psi}_M, \\ \gamma_{4E} &= \gamma_{0M}, & \gamma_{iE} &= -i\gamma_{iM}, & \gamma_{5E} &= \gamma_{5M}. \end{aligned}$$

The classical equations of motion obtained from varying the action with respect to  $\psi, \psi^\dagger$  are the Dirac eqns in the external gauge field:

$$\begin{aligned}(\gamma_\mu \nabla_\mu + im)\psi &\equiv (\not{\nabla} + im)\psi = 0, \\ \psi^\dagger(\overleftarrow{\nabla} - im) &\equiv \psi^\dagger(\gamma_\mu(\overleftarrow{\partial}_\mu + iA_\mu^a t^a) - im) = 0.\end{aligned}$$

In the massless fermion limit  $m = 0$  the fermion action in even number of dimensions is invariant under axial rotation,

$$\psi \rightarrow e^{i\alpha\gamma_5} \psi, \quad \psi^\dagger \rightarrow \psi^\dagger e^{i\alpha\gamma_5},$$

$$e^{i\alpha\gamma_5} = \mathbf{1}_2 + i\alpha\gamma_5 + \frac{1}{2!}(i\alpha\gamma_5)^2 + \dots \stackrel{\gamma_5^2=1}{=} \mathbf{1}_2 \cos \alpha + i\gamma_5 \sin \alpha.$$

The corresponding Nöther current, called the axial current,  $J_{\mu 5} \stackrel{d}{=} \psi^\dagger \gamma_m u \gamma_5 \psi$  **is conserved at the classical level**, since

$$\partial_\mu J_{\mu 5} = \psi^\dagger \overleftarrow{\nabla} \gamma_5 \psi - \psi^\dagger \gamma_5 \overrightarrow{\nabla} \psi = 2i\psi^\dagger \gamma_5 \psi \stackrel{m=0}{=} 0$$

owing to the Dirac eqns at zero  $m$ . However, Dirac eqn is a classical equation of motion, whereas the quantum fields  $\psi, \psi^\dagger$  need not be always at the saddle point.

In classical theory one differentiates  
In quantum theory one integrates

Let us consider the quantum average of  $\partial_\mu J_{\mu 5}$ :

$$\langle \partial_\mu J_{\mu 5} \rangle = \frac{\int D\psi D\psi^\dagger \left( \psi^\dagger \overleftarrow{\nabla} \gamma_5 \psi - \psi^\dagger \gamma_5 \overrightarrow{\nabla} \psi \right) e^{S_{\text{ferm}}}}{\int D\psi D\psi^\dagger e^{S_{\text{ferm}}}}, \quad S_{\text{ferm}} = \int d^d x \psi^\dagger (\overrightarrow{\nabla} + im) \psi.$$

We notice that

$$\begin{aligned} \overrightarrow{\nabla} \psi(x) e^{S_{\text{ferm}}} &= \frac{\delta}{\delta \psi^\dagger(x)} e^{S_{\text{ferm}}} \\ \psi^\dagger(x) \overleftarrow{\nabla} e^{S_{\text{ferm}}} &= -\frac{\delta}{\delta \psi(x)} e^{S_{\text{ferm}}} \end{aligned}$$

["-" in the second line is because  $\psi, \psi^\dagger$  anticommute.] We formally integrate by parts

using  $\delta\psi(y)/\delta\psi(x) = \delta(x - y)$  and obtain

$$\langle \partial_\mu J_{\mu 5} \rangle = 2\delta(0) \text{Tr} \gamma_5 = 2 \text{Sp} \gamma_5 .$$

This is an indefinite expression as  $\text{Tr} \gamma_5 = 0$  but  $\delta(0) = \infty$  ! Let us regularize this expression to see if it is zero, infinity or, maybe, a finite quantity.

$$2\text{Sp} \gamma_5 = \lim_{M \rightarrow \infty} 2 \int \frac{d^4 p}{(2\pi)^4} \text{Tr} \gamma_5 e^{-ipx} \exp\left(\frac{\not{\nabla}^2}{M^2}\right) e^{ipx} ,$$

$$\begin{aligned} \not{\nabla}^2 &= \gamma_\mu \gamma_\nu \nabla_\mu \nabla_\nu = \left(\frac{1}{2}\{\gamma_\mu \gamma_\nu\} + \frac{1}{2}[\gamma_\mu \gamma_\nu]\right) \nabla_\mu \nabla_\nu \\ &= \mathbf{1}_4 \cdot \nabla_\mu^2 - \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu}, \quad [\nabla_\mu \nabla_\nu] = -i F_{\mu\nu} . \end{aligned}$$

Tr here is the usual matrix trace but both in Dirac and YM indices. We drag  $e^{i(p \cdot x)}$  through the operator; as a result it cancels but the differentiation operator is shifted,

$\nabla_\mu \rightarrow \nabla_\mu + ip_\mu$ . We integrate over  $p$ :

$$\int \frac{d^4 p}{(2\pi)^4} \exp\left(-\frac{p^2}{M^2}\right) = \frac{M^4}{16\pi^2}.$$

Now we have to expand the exponent to the order where  $\text{Tr } \gamma_5 \dots \neq 0$ . It happens in the second order, since

$$\text{Tr } \gamma_5 \sigma_{\alpha\beta} \sigma_{\mu\nu} = -4\epsilon_{\alpha\beta\mu\nu} \neq 0$$

Here  $\epsilon_{\alpha\beta\mu\nu}$  is the 4d antisymmetric tensor,  $\epsilon_{1234} = 1$ . Expanding the exponent to the second order we get  $-F_{\alpha\beta}^a F_{\mu\nu}^a / (8M^4)$  such that  $1/M^4$  cancels with  $M^4$  from the  $p$  integration!

Therefore, the result is **finite**:

$$\langle \partial_\mu J_{\mu 5} \rangle = \frac{1}{16\pi^2} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a = \frac{1}{4\pi^2} (\mathbf{E}^a \cdot \mathbf{B}^a)$$

where  $\tilde{F}_{\mu\nu}^a \equiv \frac{1}{2}\epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}^a$  is called the dual field strength. [Electric field  $\mathbf{E}$  is dual to the magnetic field  $\mathbf{B}$  and *vice versa*.]

This is the famous **axial anomaly**: taking into account quantum fluctuations of the fermion field, the axial current is not conserved. The original symmetry of the classical theory with respect to axial rotations is explicitly broken by quantum corrections. It is **not spontaneous** breaking of symmetry, but explicit: the theory just does not have the symmetry at all.

Notice that, were fermions massive, the axial current would not be conserved even classically,

$$\partial_\mu J_{\mu 5}|_{\text{normal}} = \psi^\dagger 2im\gamma_5 \psi.$$

The anomalous piece is something one gets in addition, and it is present even in the massless limit.

I have presented probably the most economic derivation of the anomaly but there are other ways to derive it, for example, from Feynman graphs ... but it is hard work.

**History:** J. Steinberger (1949), J. Schwinger (1951), S. Adler (1969), J. Bell and R. Jackiw (1969) ... For a pedagogical review see M. Shifman, Phys. Reports 209 (1991) 341. An interesting review stressing mathematical aspects of anomalies: A. Morozov, Sov. Phys. Uspekhi 150 (1986) 337.

## Anomaly in 2d QED

To get a better insight in this remarkable phenomenon let us study a simpler case: a  $2d$  massless electrodynamics, named the **Schwinger model**:

$$S = \int d^2x \left( \frac{1}{4e^2} F_{\mu\nu}^2 + \psi^\dagger \not{\nabla} \psi \right), \quad \nabla = \partial - iA.$$

Again, there is a symmetry under axial rotations and hence a (would-be) conserved axial current,  $J_{\mu 5} = \psi^\dagger \gamma_\mu \gamma_5 \psi$ . Classically, its divergence is zero owing to the Dirac eqns of motion. However, in QFT one integrates over the fields rather than varying with respect to the fields.

We proceed exactly as in the case of  $4d$  QCD. The role of  $\gamma_5$  is played in  $2d$  by the Pauli matrix  $\sigma_3$  but I, nevertheless, denote it by  $\gamma_5$ .

$$\langle \partial_\mu J_{\mu 5} \rangle = 2\delta(0) \text{Tr} \gamma_5 = 2 \text{Sp} \gamma_5 = \lim_{M \rightarrow \infty} 2 \int \frac{d^2p}{(2\pi)^2} \text{Tr} \gamma_5 e^{-ipx} \exp \left( \frac{\not{\nabla}^2}{M^2} \right) e^{ipx},$$

$$\nabla^2 = \mathbf{1}_2 \cdot \nabla_\mu^2 - \frac{i}{2} \sigma_{\mu\nu} F_{\mu\nu}, \quad [\nabla_\mu \nabla_\nu] = -i F_{\mu\nu}.$$

We drag  $e^{i(p \cdot x)}$  through the operator; as a result it cancels but the differentiation operator is shifted,  $\nabla_\mu \rightarrow +ip_\mu$ . We integrate over  $p$ :

$$\int \frac{d^2 p}{(2\pi)^2} \exp\left(-\frac{p^2}{M^2}\right) = \frac{M^2}{4\pi}.$$

Now we have to expand the exponent to the order where  $\text{Tr} \gamma_5 \dots \neq 0$ . It happens already in the first order since

$$\text{Tr} \gamma_5 \sigma_{\mu\nu} = 2i \epsilon_{\mu\nu}.$$

$M^2$  cancels, and we obtain a finite result:

$$\langle \partial_\mu J_{\mu 5} \rangle = \frac{1}{2\pi} \epsilon_{\mu\nu} F_{\mu\nu}.$$

This is the axial anomaly for the  $2d$  QED.

## Explanation of the anomaly

We write the 2-component spinor field as

$$\begin{aligned}\psi &= \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}, \\ \psi_L &= \frac{1 + \gamma_5}{2} \psi, \quad \psi_R = \frac{1 - \gamma_5}{2} \psi, \\ \gamma_5 &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\end{aligned}$$

and call the components  $\psi_{L(R)}$  left- (right-) polarized (or 'handed') particles. The vector and axial currents are the sum and the difference of the currents of left- and right-handed particles,

$$\begin{aligned}J_\mu &= \psi_L^\dagger \gamma_\mu \psi_L + \psi_R^\dagger \gamma_\mu \psi_R, \\ J_{\mu 5} &= \psi_L^\dagger \gamma_\mu \psi_L - \psi_R^\dagger \gamma_\mu \psi_R.\end{aligned}$$

The conservation of the vector current means that the following combination does not change in time:

$$Q = \int dx J_0 = (N_L + N_R) - (N_{\bar{L}} + N_{\bar{R}}) = N - \bar{N} = \text{const.}$$

Here  $N_{L(R)}$  is the number of left- (right-) handed particles, and  $N_{\bar{L}(\bar{R})}$  is the number of left- (right-) handed antiparticles or holes, in the language of solid state physics.

The conservation of the axial current would mean that

$$Q_5 = \int dx J_{05} = (N_L - N_R) - (N_{\bar{L}} - N_{\bar{R}}) = \text{const} \quad (?)$$

is time-independent. However, the anomaly means that it is changed by the amount

$$\Delta Q_5 = \Delta ((N_L - N_R) - (N_{\bar{L}} - N_{\bar{R}})) = \frac{1}{2\pi} \int dt \int dx \epsilon_{\mu\nu} F_{\mu\nu}.$$

To see how it happens, let us choose the physical gauge  $A_0 = 0$  or, better, in the

Euclidean space,  $A_2 = 0$ . We have

$$\epsilon_{\mu\nu} F_{\mu\nu} = 2\epsilon_{\mu\nu} \partial_\mu A_\nu = -2\partial_2 A_1 = -2\dot{A}_1.$$

Therefore, we can write

$$\Delta Q_5 = -2\Delta \left( \int dx \frac{A_1}{2\pi} \right).$$

Let us consider a simple example: we put a system in a box  $[0, L]$  and change  $A_1$  adiabatically in time starting from  $A_1 = 0$ . Let it remain a constant in the space interval  $[0, L]$ . We see that when it reaches the value of  $A_1 = \frac{2\pi}{L}$  the axial charge of the fermion system changes by two units! How can it happen?

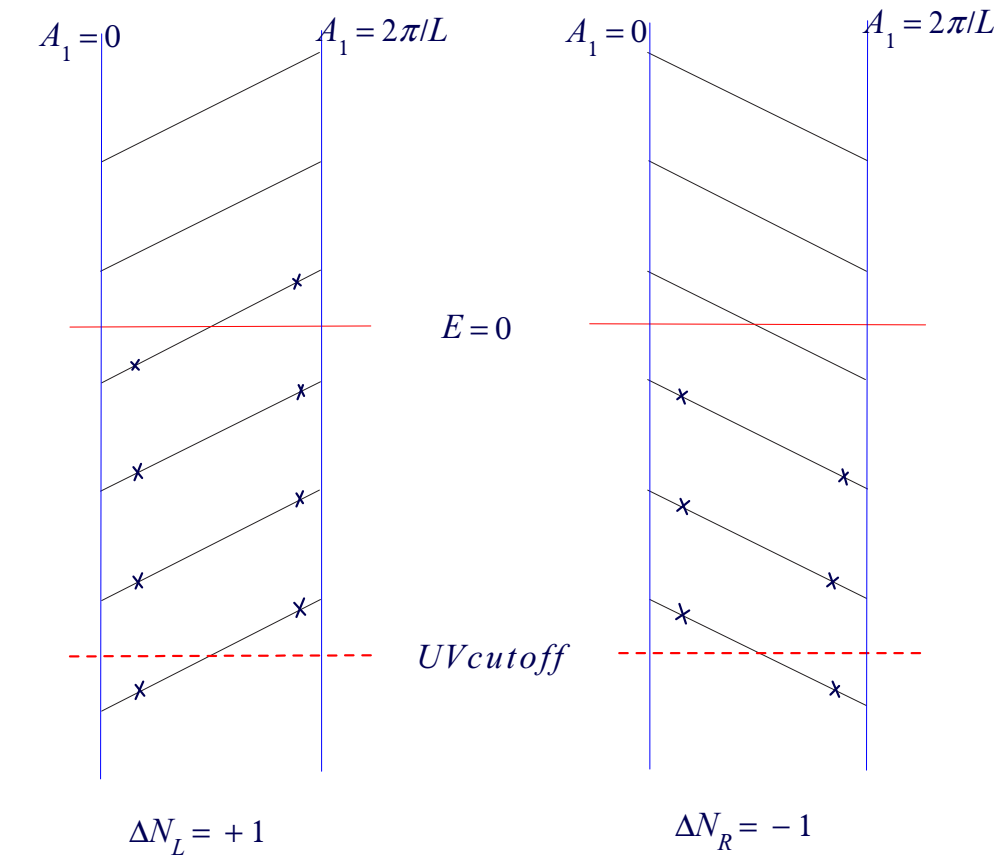
It is easy to solve the Dirac equation in a such a simple case.

$$\left[ \gamma_2 \frac{\partial}{\partial t} + \gamma_1 \left( \frac{\partial}{\partial x} - iA_1 \right) \right] \psi = 0, \quad \psi(x, t) = e^{-iEt} \phi(x),$$
$$\sigma_3 \left( i \frac{\partial}{\partial x} + A_1 \right) \phi(x) = E\phi(x). \quad \text{stationary equation for energy levels.}$$

This is a two-component eqn: they differ by the sign of  $E$ . We impose an anti-periodic boundary condition for the fermion wave functions,  $\phi(L) = -\phi(0)$ . For the upper component (the left-handed fermions) the wave functions are

$$\phi_L(x) = \exp \left[ i \left( n + \frac{1}{2} \right) \frac{2\pi}{L} x \right], \quad E_{L n} = - \left( n + \frac{1}{2} \right) \frac{2\pi}{L} + A_1.$$

For the lower component (the right-handed fermions) the wave function is the same but the energy has the opposite sign (for given integer)  $n$ . The spectrum is equidistant, it stretches from  $-\infty$  to  $+\infty$ , and the levels move adiabatically as we slowly change  $A_1$  from 0 to  $\frac{2\pi}{L}$ .



Energy levels of left-handed fermions

Energy levels of right-handed fermions

$$\Delta Q = \Delta N_L + \Delta N_R = 0$$

$$\Delta Q_5 = \Delta N_L - \Delta N_R = 2$$

According to the Dirac theory the vacuum corresponds to filling in all states with negative energies, both for left-handed and right-handed fermions. When  $A_1$  reaches the value of  $\frac{2\pi}{L}$  all energy levels assume their original positions, however their fillings change! A right-handed fermion is now having a positive energy, while a left-handed fermion goes down and leaves one level with negative energy unoccupied. If we compare with the original vacuum at  $A_1 = 0$ , one right-handed fermion is created and one left-handed one is annihilated! Note that the vector charge is conserved but the axial charge is changed exactly by two units, as predicted by the anomaly!

The crucial detail in this story [V. Gribov (1981)] is that we are counting states only above certain threshold, the cutoff. A left-handed fermion has dipped under the cutoff while a right-handed fermion emerged from under the cutoff. Without an explicit UV regularization we would not be able to see the effect, but QFT's are not defined without a regularization.

Another important detail is that a normal-looking field (here:  $A_1$ ) affects very deep-lying levels. It is counter-intuitive: usually particles with very high momenta do not 'notice' the external disturbances and move as free. However, in some specific cases particles with any momenta are affected by the presence of the external field: when the Dirac sea moves as a whole as one adiabatically changes the external field – this is the case of the anomaly.

Quantum anomaly has two faces: the “ultraviolet face” and the “infrared” one. On the one hand one needs an UV regularization to treat the problem accurately. On the other hand, one observes the appearance and disappearance of low-momenta levels. This is because the Dirac sea moves as a whole. The IR side of the anomaly has an important consequences related to fermion zero modes. That topic will be studied in L-10.

## Interpretation of the axial anomaly in QCD

The interpretation of the axial current in  $4d$  theory is the same as in the  $2d$  Schwinger model.

$$Q_5 = \int d^3x J_{05} = (N_L - N_R) - (N_{\bar{L}} - N_{\bar{R}})$$

is the axial charge of the fermion system. Classically, it is conserved but quantum effects lead to its non-conservation. One can integrate over full space and time the anomaly equation for  $\partial_\mu J_{\mu 5}$  and obtain

$$\Delta Q_5 = \frac{1}{16\pi^2} \int dt \int d^3x F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a.$$

An important point is that the integrand is a full derivative:

$$\begin{aligned} \frac{1}{16\pi^2} F_{\mu\nu}^a \tilde{F}_{\mu\nu}^a &= 2\partial_\mu K_\mu, \\ K_\mu &= \frac{1}{16\pi^2} \epsilon_{\mu\alpha\beta\gamma} \left( A_\alpha^a \partial_\beta A_\gamma^a + \frac{1}{3} f^{abc} A_\alpha^a A_\beta^b A_\gamma^c \right). \end{aligned}$$

Assuming that the YM field  $A_\mu^a$  is decaying fast enough at spatial infinity we obtain

$$\Delta Q_5 = 2 \int_{-\infty}^{+\infty} dt \frac{dN_{\text{CS}}}{dt} = 2\Delta N_{\text{CS}}, \quad N_{\text{CS}} \stackrel{d}{=} \int d^3x K_0.$$

The functional  $N_{\text{CS}}[A]$  is a famous quantity: it is called the **Chern–Simons** number. It has the following remarkable property: Let us perform a large gauge transformation with the help of a time-independent unitary matrix  $S(x)$ :

$$A_i \rightarrow S^\dagger A_i S + iS^\dagger \partial_i S$$

with  $S(x)$  tending to the unity matrix at spatial infinity. Then the Chern–Simons number shifts by

$$N_{\text{CS}} \rightarrow N_{\text{CS}} + \frac{1}{24\pi^2} \int d^3\mathbf{x} \epsilon_{ijk} \text{Tr} \left( S^\dagger \partial_i S \right) \left( S^\dagger \partial_j S \right) \left( S^\dagger \partial_k S \right).$$

This integral gives the winding number of the mapping  $S^3 \rightarrow S^3$  determined by  $S(x)$ . Therefore, under gauge transformations the Chern–Simons number either does not change

at all (if the mapping  $S(x)$  can be continuously deformed to  $S(x) = 1$ ) or is shifted by an integer.

Imagine, we start from  $A_i = 0$  and adiabatically change it with time in such a way that finally we reach a field which is a “pure gauge”,

$$A_i = iS^\dagger \partial_i S$$

with the unitary matrix  $S(x)$  having  $N_{\text{wind}} = 1$ . It means that we change  $N_{\text{CS}}$  from 0 to 1, such that  $\Delta N_{\text{CS}} = 1$ . The anomaly says that, in such a case,  $\Delta Q_5 = 2$ : one left-handed fermion has been created and one right-handed fermion has been annihilated during this process, see the Fig. above.

An interesting quantum anomaly in super  $^3\text{He-A}$  and  $^3\text{He-B}$  has been found theoretically and confirmed experimentally – see

V. Eltsov, M. Krusius and G. Volovik, “Super  $^3\text{He}$ : a Laboratory Model System for Quantum Field Theory”, e-print cond-mat/9809125.