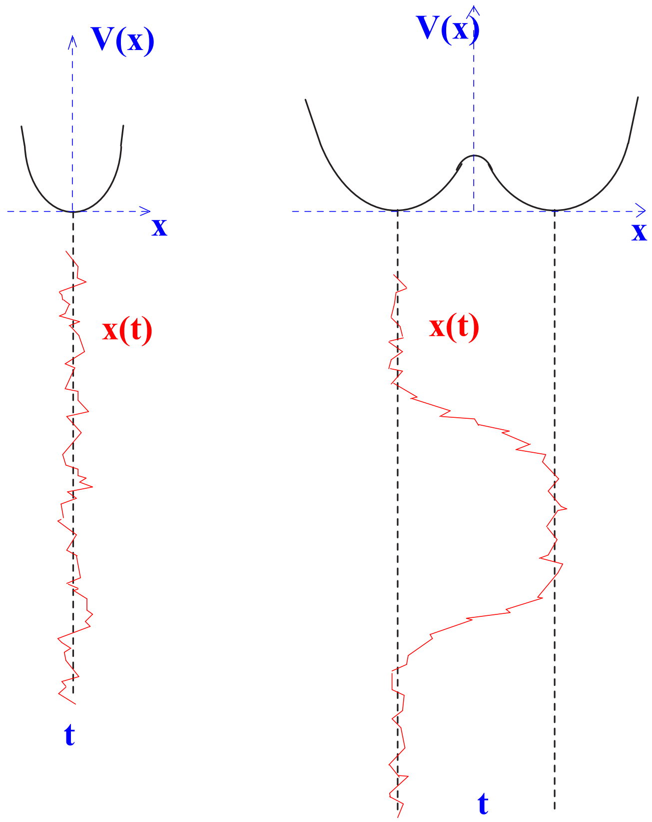
Yang–Mills theory in 2,3,4 dimensions on the lattice

"Vacuum" state in $0+1$ dimensional field theory (i.e. Quantum Mechanics)

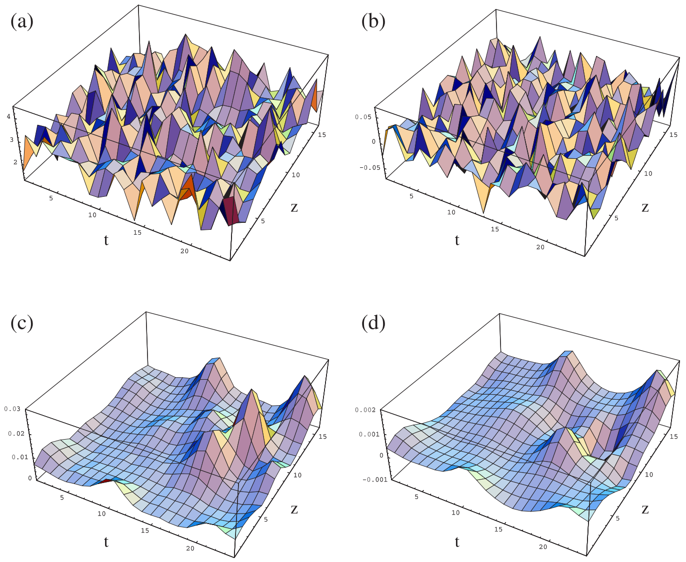


This comes out from the path integral

$$\mathcal{Z} = \int Dx(\tau)\, \exp\left\{-\int_0^T d\tau\, \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V\left(x(\tau)\right)\right]\right\}$$

whose discretized (lattice) version is

$$\mathcal{Z} = \prod_n \int dx_n\, \exp\left\{-\sum_n a\left[\frac{m}{2}\left(\frac{x_n - x_{n-1}}{a}\right)^2 + V(x_n)\right]\right\}.$$

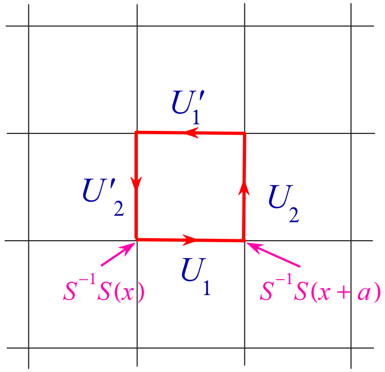


Comes out from the Yang–Mills path integral invariant under $A_\mu \to S A_\mu S^{-1} + iS\partial_\mu S^{-1}$:

$$\mathcal{Z} = \int DA_\mu(x)\, \exp\left(-\frac{1}{4g^2}\int d^4x\, F^a_{\mu\nu}F^a_{\mu\nu}\right)$$

whose discretized (lattice) version is

$$\mathcal{Z} = \prod_{\text{links}} \int dU_{\text{link}}\, \exp\left\{-\sum_{\text{plaq}} \beta\, \mathrm{Tr}\, U_{\text{plaq}}\right\}.$$



$$U_{\text{plaquette}} \overset{d}{=} U_1 U_2 U'_1 U'_2$$

is gauge invariant. Link variables transform under gauge transformation as

$$U_{\text{link}} \to S(x)\, U_{\text{link}}(x \to x+a) S^{-1}(x+a)$$

$$U^{\text{link}}_\mu = \exp\left[i\, a\, t^a\, A^a_\mu\, (x + a/2)\right]$$

YM theory on the lattice [K. Wilson (1974), A. Polyakov (1974)]

Lattice-regularized partition function

$$
\begin{aligned}
\mathcal{Z}(\beta) &= \int \prod_{\rm links} dU_{\rm link}\ \exp\left(\sum_{\rm plaq} \beta\, \frac{\mathsf{Tr}\ U_{\rm plaq} + {\rm c.c.}}{2\ \mathsf{Tr}\ 1}\right) \\
&\to \int DA_\mu\ \exp\left(-\frac{1}{2g_d^2}\int d^dx\ \mathsf{Tr}\ F_{\mu\nu}^2\right), \\
\beta &= \frac{2N}{a^{4-d} g_d^2}
\end{aligned}
$$

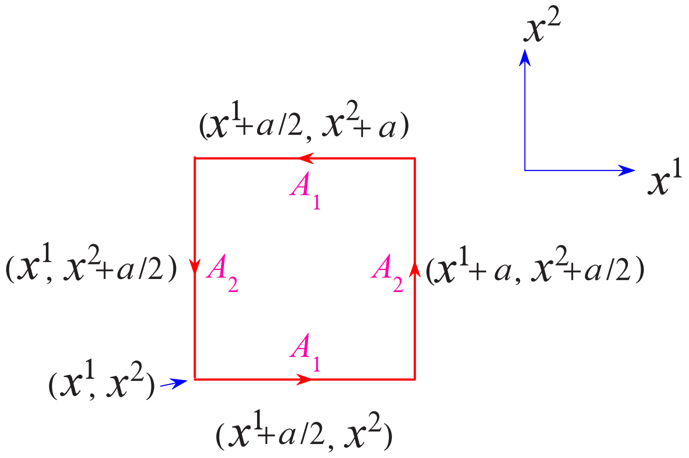
To pass to the continuum limit, one writes the unitary matrix belonging to a lattice link pointing in direction μ $(\mu = 1 \ldots d)$

$$
U_\mu^{\rm link} = \exp\left[i\, a\, t^a\, A_\mu^a\left(x + \frac{a}{2}\right)\right]
$$

and expands Tr $(U_\mu U_\nu U'_\mu U'_\nu)$ in a. The first non-zero term is $\mathcal{O}(a^4)$.

For example, to evaluate Tr U_{12} lying in the $(1,2)$ plane, one has to expand

$$\mathrm{Tr}\,U_{12} = \mathrm{Tr}\Bigg(\exp\left[iaA_1\left(x^1+\frac{a}{2},x^2\right)\right]\cdot\exp\left[iaA_2\left(x^1+a,x^2+\frac{a}{2}\right)\right] \cdot\exp\left[-iaA_1\left(x^1+\frac{a}{2},x^2+a\right)\right]\cdot\exp\left[-iaA_2\left(x^1,x^2+\frac{a}{2}\right)\right]\Bigg)$$



$$\frac{\mathrm{Tr}\;U_{12}+\mathrm{c.c.}}{2\,\mathrm{Tr}\;1} = 1-\frac{a^4}{2N}\,\mathrm{Tr}\,F_{12}F_{12}+\mathcal{O}(a^6)$$

$$F_{12}=\partial_1A_2-\partial_2A_1-i[A_1A_2]$$

[Do it!]

At each lattice site there are $\frac{d(d-1)}{2}$ plaquettes in the positive direction, over which one has to sum in $\sum_{\text{plaq}}$, and one has to sum over all sites of the lattice. That gives

$$-\sum_{\text{sites}} a^d \left(\frac{\beta\, a^{4-d}}{4N}\,\mathsf{Tr}\, F_{\mu\nu}F_{\mu\nu} + \mathcal{O}(a^{6-d})\right)$$

$$\to -\int d^d x \left(\frac{\beta\, a^{4-d}}{4N}\,\mathsf{Tr}\, F_{\mu\nu}F_{\mu\nu} + \mathcal{O}(a^{6-d})\right) = -\frac{1}{2g_d^2}\int d^d x\ \mathsf{Tr}\ F_{\mu\nu}F_{\mu\nu} + \ldots$$

$$\Longrightarrow \beta = \frac{2N}{a^{4-d} g_d^2}\,.$$

The continuum limit is obtained at $a \to 0, \quad \beta \to \infty$ and

$$\left\{\begin{array}{rl} g_2^2 = \frac{2N}{a^2\beta} = \text{fixed}, & d = 2, \\ g_3^2 = \frac{2N}{a\beta} = \text{fixed}, & d = 3, \\ \Lambda = \frac{1}{a}\,\exp\left(-\frac{12\,\beta\,\pi^2}{11\,N^2}\right) = \text{fixed}, & d = 4. \end{array}\right.$$

Λ has the dimension on mass and gives the scale in the continuum theory [Dimensional transmutation]. It is known from experiment. YM theory in $4d$ is asymptotically free, with the 'running coupling constant' given by

$$\frac{8\pi^2}{g^2(\mu)} = \frac{11}{3} N \, \ln\frac{\mu}{\Lambda}, \qquad \Lambda = \mu \, \left(=\frac{1}{a}\right) \cdot \exp\left(-\frac{8\pi^2}{\frac{11}{3} N \, g^2}\right) .$$

Observables: gauge-invariant Wilson loops

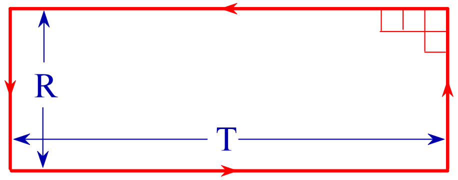
$$W_J[C] = \frac{1}{2J+1} \mathrm{Tr}\, \mathrm{P} \, \exp \, i \oint_C dx^\mu \, A^a_\mu \, t^a_J \quad \to \quad \frac{1}{2J+1} \mathrm{Tr}_J \left(\prod_{\text{links}} U_J^{\text{link}} \ldots \right) .$$

Expect

- $<$large Wilson loop$> \sim \exp(-m^2 \cdot \text{Area})$ if J is half-integer (N-ality non-zero)
- $< W(x) W(y) > \sim \exp(-m \cdot \text{Separation})$

mass: $m \sim g_2$ $(2d)$, $\quad \sim g_3^2$ $(3d)$, $\quad \sim \Lambda = \frac{1}{a} \exp\left(-\frac{8\pi^2}{11\, g_4^2}\right)$ $(4d)$

Wilson's criteria of confinement: the area behaviour of the Wilson loop



$$W_J = \prod_{\text{links}} \mathrm{Tr}_J(D^J(U)D^J(U).....) \to \mathrm{Tr}\,\mathrm{P}\,\exp i \oint dx^\mu\, A^a_\mu\, t^a_J$$

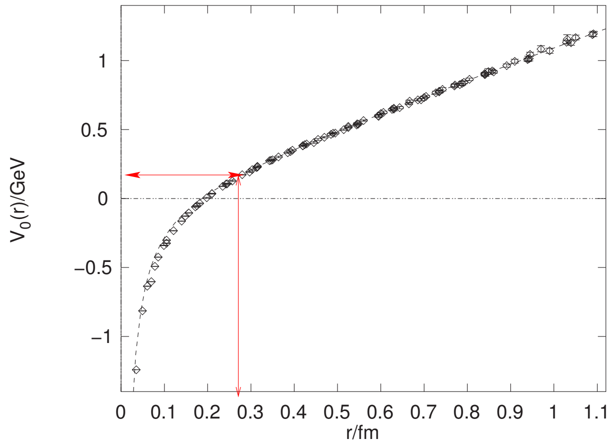
Average $\langle W \rangle = \exp\left[-V(R)\,T\right]$ at $T \to \infty$.

$V(R)$ is the potential between quark and antiquark at separation R.

The area law, $W \sim \exp(-\sigma\, \mathrm{Area})$, means the linear rising potential:

$$V(R) = \sigma\, R, \qquad \text{string tension } \sigma \simeq \begin{cases} g_2^2 & \text{in} \quad d = 2, \\ g_3^4 & \text{in} \quad d = 3, \\ \Lambda^2 & \text{in} \quad d = 4. \end{cases}$$

Expected: non-zero and equal σ for all half-integer representations J, and asymptotically zero σ for all integer J's.



The potential energy of two infinitely-heavy quarks, as function of their separation, simulated on an $SU(2)$ lattice. The units come from setting $\sqrt{\sigma} = 420\,\mathrm{MeV}$ [G. Bali et al. (1995)].

Mass gap

Correlation function

$$< \mathrm{Tr}U(x)\,\mathrm{Tr}U(y) > \sim \exp(-m\,|x-y|) \qquad \left(\simeq < F^a_{\mu\nu}F^a_{\mu\nu}(x)\,F^b_{\kappa\lambda}F^b_{\kappa\lambda}(y) > \right)$$

$$m = c_m\,\Lambda = \frac{1}{a}\exp\left(-\frac{8\pi^2}{\frac{11}{3}N\,g^2}\right).$$

Numerically, $m \simeq 1400\,\mathrm{MeV}$.

Integration over the invariant group measure (Haar measure)

One has to integrate over $SU(N)$ matrices U (parametrized by N^2-1 parameters, called 'Euler angles'. The integration measure must be invariant under multiplication of U by a constant unitary matrix V both from the *left* and from the *right*:

$$\int dU = \int d(UV) = \int d(VU) \qquad \left[\text{similar to} \quad \int dx = \int d(x+\text{const})\right]$$

One can define the group integrals abstractly, without writing explicitly matrix parametrization and the explicit measure for the Euler angles:

$$\int dU = 1, \qquad\qquad \int dU\, U_i^\alpha U_\beta^{\dagger j} = \frac{1}{N}\delta_\beta^\alpha \delta_i^j$$

$$\int dU\, U_i^\alpha U_j^\beta = \begin{cases} \frac{1}{2}\,\epsilon_{ij}\,\epsilon^{\alpha\beta} & \text{for } SU(2) \\ 0 & \text{for } SU(N),\ N>2 \end{cases}$$

$$\int dU\, U^{\alpha_1}_{i_1} U^{\alpha_2}_{i_2} U^{\dagger j_1}_{\beta_1} U^{\dagger j_2}_{\beta_2}$$
$$= \frac{1}{N^2-1}\left[\delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}\left(\delta^{j_1}_{i_1}\delta^{j_2}_{i_2} - \frac{1}{N}\delta^{j_1}_{i_2}\delta^{j_2}_{i_1}\right) + \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_1}\left(\delta^{j_1}_{i_2}\delta^{j_2}_{i_1} - \frac{1}{N}\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}\right)\right],$$
$$\ldots$$

Explicit parametrization of $SU(N)$ matrices by N^2-1 ‘Euler angles’

$SU(2)$, 3 parameters

$$U_2 = \begin{pmatrix} e^{-i\alpha_{11}}\cos\phi_1 & e^{i\alpha_{12}}\sin\phi_1 \\ -e^{-i\alpha_{12}}\sin\phi_1 & e^{i\alpha_{11}}\cos\phi_1 \end{pmatrix}$$

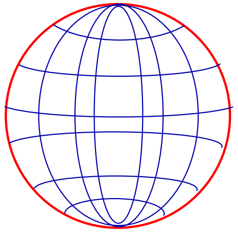
where the last column can be viewed as a $2d$ complex vector $v_2 = (z^1, z^2)$ normalized as $|z^1|^2 + |z^2|^2 = 1$, which defines an S^3 sphere. The first column is the orthogonal vector $v^i_1 = \epsilon^{ij}\bar{v}_{2j}$.

The group measure can be written as an integral over the S^3 sphere,

$$\frac{1}{\pi^2}\int dz^1 d\bar{z}^1 dz^2 d\bar{z}^2\, \delta(|z^1|^2+|z^2|^2-1),$$

or, explicitly in terms of three angles, as

$$\frac{1}{2\pi^2}\int_0^{\frac{\pi}{2}} d\phi_1\ \sin\phi_1 \cos\phi_1 \int_0^{2\pi} d\alpha_{11}\int_0^{2\pi} d\alpha_{12} \qquad (=1)$$



3-sphere $S^3 : \phi, \alpha_1, \alpha_2$

Exercise

Using the explicit parametrization and the explicit Haar measure, check the above general relations for $\int dU U^\dagger U$, $\int dU U^\dagger U U^\dagger U$.

$SU(3)$, 8 parameters

We build the parametrization iteratively:

$$R_2 = \left(\begin{array}{ccc} \multicolumn{2}{c}{\multirow{2}{*}{U_2}} & 0 \\ & & 0 \\ 0 & 0 & 1 \end{array} \right),$$

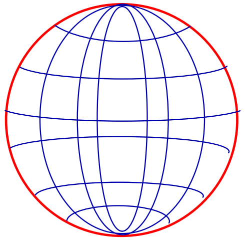
and define

$$U_3 = S_3 R_2, \qquad S_3 = \left(\begin{array}{ccc} e^{i\alpha_{23}} \cos\theta & 0 & e^{i\alpha_{23}} \sin\theta \\ -e^{i\alpha_{22}} \sin\theta \sin\phi_2 & e^{-i\alpha_{21}-i\alpha_{23}} \cos\phi_2 & e^{i\alpha_{22}} \cos\theta \sin\phi_2 \\ -e^{i\alpha_{21}} \sin\theta \cos\phi_2 & -e^{-i\alpha_{22}-i\alpha_{23}} \sin\phi_2 & e^{i\alpha_{21}} \cos\theta \cos\phi_2 \end{array} \right).$$

The last column can be viewed as a $3d$ complex vector $v_3 = (z^1, z^2, z^3)$ normalized to $|z^1|^2 + |z^2|^2 + |z^3|^2 = 1$, which defines an S^5 sphere. The three columns are constructed as (complexified) orts in spherical coordinates: $v_1 \sim e_r$, $v_2 \sim e_\phi$, $v_3 \sim e_\theta$. We use part of the freedom of choosing the orts and the angles in such a way that $U_3 = \mathbf{1_3}$ when all angles are set to zero.

The measure on S^5 can be written as

$$\frac{2}{\pi^3}\int dz^1 d\bar{z}^1 dz^2 d\bar{z}^2 dz^3 d\bar{z}^3\,\delta(|z^1|^2+|z^2|^2+|z^3|^2-1)$$



5-sphere S^5 : $\theta,\phi,\alpha_1,\alpha_2,\alpha_3$

or, explicitly in terms of five angles, as

$$\frac{1}{\pi^3}\int_0^{\frac{\pi}{2}} d\theta\ \cos^3\theta\sin\theta\int_0^{\frac{\pi}{2}} d\phi_2\ \sin\phi_2\cos\phi_2\int_0^{2\pi} d\alpha_{21}\int_0^{2\pi} d\alpha_{22}\int_0^{2\pi} d\alpha_{23}\qquad (=1).$$

The integrations limits are chosen such that the S^5 sphere is covered once.

The full $SU(3)$ Haar measure is found in the following way

1. One constructs the *vielbein*

$$e^a_m = -i\mathrm{Tr}\left(U^\dagger \frac{\partial U}{\partial \beta_m} t^a\right), \qquad \beta^m = \alpha_{11}, \alpha_{12}, \phi_1, \alpha_{21}, \alpha_{22}, \alpha_{23}, \phi_2, \theta, \quad m, n = 1...8,$$

such that the metric tensor is

$$g_{mn} = e^a_m e^a_n = \mathrm{Tr}\,\frac{\partial U^\dagger}{\partial \beta^m}\frac{\partial U}{\partial \beta^n}, \qquad U = S_3 R_2.$$

The Haar integration measure is (as usual, if one knows the metrics)

$$\int d\beta_1 \ldots d\beta_8 \sqrt{\det\ g_{mn}},$$

$$\det g_{mn} = \det\left(e^a_m\, e^a_n\right) = \det\left(e \cdot e^T\right) = \det e \cdot \det e^T = (\det\ e)^2\,.$$

2. The vielbein appears to be a block-triangle matrix:

$$U^\dagger \frac{\partial U}{\partial \beta_m} = R_2^\dagger S_3^\dagger \frac{\partial}{\partial \beta_m}(S_3 R_2) = R_2^\dagger \left(S_3^\dagger \frac{\partial S_3}{\partial \beta_m}\right) R_2 + R_2^\dagger \frac{\partial R_2}{\partial \beta_m}.$$

The second term is nonzero for $m = 1,2,3$, the second term is nonzero for $m = 4,5,6,7,8$. The second term has a nonzero projection to $t^{a=1,2,3}$ only, hence $e_m^a = 0$ for $m = 1,2,3$ and $a = 4,5,6,7$ Therefore, e_m^a understood as an 8×8 matrix, is block-triangle.

3. The determinant of a block-triangle matrix is a product of the determinants of the diagonal blocks:

$$\det e = \det e_{3\times 3} \cdot \det e_{5\times 5}$$

4. Therefore, the Haar integration measure of the $SU(3)$ group is a product of integration measures over the spheres S^3, S^5:

$$\sqrt{\det g_{SU(3)}} = \sqrt{\det g_{S^3}} \cdot \sqrt{\det g_{S^5}} \quad !$$

5. By induction, one proves for a general $SU(N)$ group that the invariant Haar integration measure over the group is a product of integration measures over **odd-dimensional** spheres:

$$\sqrt{\det g_{SU(N)}} = \sqrt{\det g_{S^3}} \cdot \ldots \cdot \sqrt{\det g_{S^{2N-1}}} \quad !$$

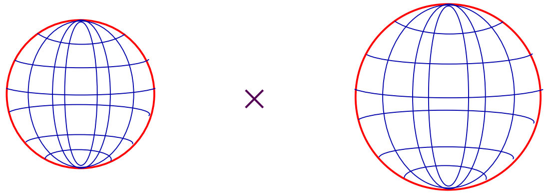
In particular, the $SU(3)$ Haar measure is

$$\sqrt{\det g} = (\sin\phi_1 \cos\phi_1) \cdot (\cos^3\theta \sin\theta \sin\phi_2 \cos\phi_2)$$

$i.e.$ it is **factorized** into the product of the measures over the spheres S^3 and S^5.

Conclusions

A general $SU(3)$ matrix can be written through 8 "Euler angles" parameterizing the $S^3 \times S^5$ spheres



3-sphere S^3: $\phi_1, \alpha_{11}, \alpha_{12}$

5-sphere S^5: $\theta, \phi_2, \alpha_{21}, \alpha_{22}, \alpha_{23}$

In general, $SU(N)$ can be parameterized by N^2-1 "Euler angles" on the odd-dimensional spheres: $S^3 \times S^5 \times \ldots \times S^{2N-1}$.

Dual transformation on the lattice, $SU(2)$

The idea: Link variables are not gauge invariant: the plaquette variables are. Cannot we reformulate the theory completely in terms of gauge invariant plaquette variables?
Insert a unity for every plaquette:

$$1 \quad = \quad \prod_{\text{plaquettes}} \int dU_{\text{plaq}} \delta(U_{\text{plaq}}\,,\; U_1U_2U_3^\dagger U_4^\dagger)$$

$$\delta(U,V) \quad = \quad \sum_{J=0,\frac{1}{2},1,\frac{3}{2},\ldots} (2J+1)\; D^J_{m_1m_2}(U^\dagger)D^J_{m_2m_1}(V),$$

$$V \quad = \quad U_1U_2U_3^\dagger U_4^\dagger. \qquad \left(\text{analogue of} \quad \delta(x-y) = \int \frac{dp}{2\pi}\, e^{ipx}\, e^{-ipy}.\right)$$

$D^J_{mn}(\alpha,\beta,\gamma)$ are Wigner finite-rotation matrices and depend on Euler angles α,β,γ.

They are $(2J+1)^2$-fold degenerate eigenfunctions of the angular momentum operator,

$$\begin{aligned} \mathbf{J}^2\, D^J_{mn} &= J(J+1)\, D^J_{mn} \\ J &= 0,\, \frac{1}{2},\, 1,\, \frac{3}{2},\, \ldots \\ -J \le m &\;,\; n \le J. \end{aligned}$$

They are ortho-normalized,

$$\int dU\, D^{J_1}_{kl}(U^\dagger)\, D^{J_2}_{mn}(U) = \frac{1}{2J+1}\delta_{J_1 J_2}\,\delta_{kn}\,\delta_{lm}.$$

Integration over plaquette variables factorizes into:

$$\int dU \exp\left(\beta \frac{\mathrm{Tr}\, U + \mathrm{Tr}\, U^\dagger}{2\,\mathrm{Tr}\, 1}\right) D^J_{mn}(U^\dagger)$$

$$= \delta_{mn}\, \frac{2}{\beta} I_{2J+1}(\beta)$$

$$\frac{2}{\beta} I_{2J+1}(\beta) = \frac{2}{\beta} I_1(\beta)\ T_J(\beta),$$

$$T_J(\beta) \to \exp\left[-\frac{2J(J+1)}{\beta}\right]$$

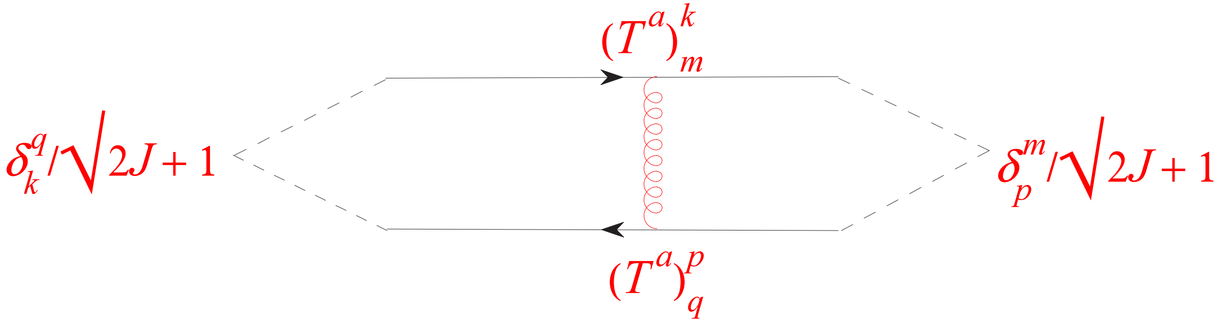
NB: continuum limit: $J \sim \sqrt{\beta} \gg 1$!

In $2d$ because of the orthogonality of D-functions all plaquettes have the same J outside the loop and the same J' inside the loop, with $|J - j_s| \le J' \le J + j_s$ where j_s is the 'color spin' of the source along the loop. The average Wilson loop is, therefore, exactly computable in $2d$:

$$\langle W_{j_s}(S)\rangle \quad = \quad \frac{\sum_J \left[T_J(\beta)\right]^{\frac{V}{a^2}} \sum_{J'=|J-j_S|}^{J+j_S} \left[T_{J'}(\beta)\,/\,T_J(\beta)\right]^{\frac{S}{a^2}}}{\sum_J \left[T_J(\beta)\right]^{\frac{V}{a^2}}}$$

$$\to \quad [T_{j_s}(\beta)]^{\frac{S}{a^2}} \to \exp\left[-\frac{g_2^2}{2} j_s(j_s+1)\; S\right],$$

– the needed area behavior (with the ‘Casimir’ string tension) $\Rightarrow$ confinement in $d = 1{+}1$!



$$\mathrm{Tr}(T^a\, T^a) = J(J+1)(2J+1)$$

$$-\frac{\partial^2}{\partial\, x^2}\phi = g^2\delta(x) \Rightarrow \phi = \frac{g^2}{2}\,|x|$$

$$V = \frac{g^2}{2} J(J+1)\,|x-y|$$

is the Coulomb energy of a quark and an antiquark with ‘isospin’ J in $2d$.

3jm symbols (≈ Clebsch–Gordan coeff's):

$$\int dU\ D^{J_1}_{a_1 b_1}(U)\ D^{J_2}_{a_2 b_2}(U)\ D^{J_3}_{a_3 b_3}(U)$$

$$= \begin{pmatrix} J_1 & J_2 & J_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} J_1 & J_2 & J_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$

6j symbols = a contraction of four 3jm's:

$$\sum_{klmnop} (-1)^{j_4+n+j_5+o+j_6+p} \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_1 & j_5 & j_6 \\ k & o & -p \end{pmatrix}$$

$$\times \begin{pmatrix} j_4 & j_2 & j_6 \\ -n & l & p \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_3 \\ n & -o & m \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$$

9j symbols = a contraction of six 3jm's:

$$\sum \begin{pmatrix} j_1 & j_2 & j_3 \\ k & l & m \end{pmatrix} \begin{pmatrix} j_4 & j_5 & j_6 \\ n & o & p \end{pmatrix} \begin{pmatrix} j_7 & j_8 & j_9 \\ q & r & s \end{pmatrix} \begin{pmatrix} j_1 & j_4 & j_7 \\ k & n & q \end{pmatrix}$$

$$\times \begin{pmatrix} j_2 & j_5 & j_8 \\ l & o & r \end{pmatrix} \begin{pmatrix} j_3 & j_6 & j_9 \\ m & p & s \end{pmatrix} = \begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}$$