Yang–Mills theory in 2,3,4 dimensions

It has been noticed that G-d has not been tremendously inventive: having once discovered successful dynamics He tends to use it again and again under various circumstances. One of such dynamical systems is, undoubtedly, quantum YM theory. Apart from being the basis of the Standard Model of fundamental interactions, non-Abelian gauge symmetry is found in condensed matter systems, e.g. in superfluid $^3\mathrm{He}$ [see G. Volovik et al., cond-mat/9809125]. Quantum Gravity can be also thought of as a non-Abelian gauge theory.

According to the study by Jackson and Okun into the history [Rev. Mod. Phys. **73** (2001) 663, `hep-ph/0012061`], gauge symmetry in classical electrodynamics was discovered by a Danish physicist Ludvig Lorenz (1840's), and in quantum mechanics by Vladimir Fock [Zeit. f. Physik (1926)]. The term 'gauge invariance' has been introduced by Hermann Weyl (1929). The non-Abelian gauge symmetry was invented by Yang and Mills (1954), although there are rumors that Pauli new about it long before but didn't publish his work as he (correctly) thought there were no massless gauge fields in our world...

$SU(N)$ gauge group

YM theory is based on semi-simple continuous Lie groups. We'll restrict ourselves to a particular case of $SU(N)$ groups, which is sufficient for most practical purposes. $SU(N)$ groups are formed by unitary ($UU^\dagger = U^\dagger U = \mathbf{1}$) unimodular ($\det U = 1$) $N \times N$ matrices having $N^2 - 1$ degrees of freedom, i.e. described by $N^2 - 1$ functions of d variables where d is the dimension of space-time. A possible parametrization of $SU(N)$ group elements is

$$U(x) = e^{i\omega^a(x)\, t^a}$$

where t^a are $N^2 - 1$ traceless hermitian $N \times N$ matrices. We shall assume the normalization

$$\mathrm{Tr}\, t^a t^b = \tfrac{1}{2}\delta^{ab}, \qquad t^a t^a = \frac{N^2-1}{2N}\,\mathbf{1}.$$

For example, in $SU(2)$ one can choose the basis of Pauli matrices,

$$t^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad t^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In $SU(3)$ one can choose 8 Gell-Mann matrices,

$$\begin{aligned}
t^1 &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
t^3 &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad t^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\
t^5 &= \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad t^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},
\end{aligned}$$

$$t^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad t^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The group generators t^a close under commutation relations,

$$\left[t^a\, t^b\right] = i\, f^{abc}\, t^c$$

where the totally antisymmetric f^{abc} are called the structure constants of the group. For the $SU(N)$ group

$$f^{abc}\, f^{abd} = N\, \delta^{cd}.$$

In $SU(2)$ $f^{abc} = \epsilon^{abc}$.

The fundamental representation of $SU(N)$ is formed by N-spinors; they transform by the $N \times N$ matrices:

$$\psi^\alpha \to U^\alpha_\beta\, \psi^\beta, \qquad \alpha, \beta = 1 \ldots N.$$

Complex-conjugate N-spinors (which should be presented by a row) transform by the hermitian-conjugated matrix,

$$\psi_\alpha^\dagger \to \psi_\beta^\dagger \, U_\alpha^{\dagger\,\beta}.$$

It is clear that the combination $\psi_\alpha^\dagger \psi^\alpha$ is an $SU(N)$ invariant,

$$\psi_\alpha^\dagger \psi^\alpha \to \psi_\beta^\dagger \, U_\alpha^{\dagger\,\beta} \, U_\gamma^\alpha \, \psi^\gamma = \psi^\dagger \psi.$$

There are also quantities transforming according to the $N^2 - 1$-dimensional adjoint representation,

$$\begin{aligned} B^a &\to O^{ab} \, B^b, \qquad O^{ab} = \tfrac{1}{2}\mathrm{Tr}(U^\dagger t^a U t^b), \\ O^{ab} O^{ac} &= \delta^{bc}. \end{aligned}$$

To verify the last eqn. you'll need the Fiertz identity,

$$(t^a)^\alpha_\beta \, (t^a)^\gamma_\delta = -\frac{1}{2N} \delta^\alpha_\beta \, \delta^\gamma_\delta + \frac{1}{2} \delta^\alpha_\delta \, \delta^\gamma_\beta$$

(summation over a is understood here). Please check that if a quantity B^a transforms according to the adjoint representation the following two combinations are invariants of the $SU(N)$ rotations:

$$B^a B^a = \mathrm{invariant}, \qquad B^a\,\psi^\dagger\,t^a\,\psi = \mathrm{invariant}.$$

Another useful "antisymmetric Fiertz identity":

$$f^{abc}\,(t^a)^\alpha_\beta\,\left(t^b\right)^\gamma_\delta = \frac{i}{2}\left[(t^c)^\alpha_\delta\;\delta^\gamma_\beta - (t^c)^\gamma_\beta\;\delta^\alpha_\delta\right].$$

In all Lie algebras higher than $SU(2)$ one can introduce *symmetric* analogs of f^{abc} defined as

$$f^{abc} \;=\; -2i\,\mathrm{Tr}(t^a t^b - t^b t^a)t^c, \qquad f^{abc}f^{abd} = N\,\delta^{cd},$$

$$d^{abc} \;=\; 2\,\mathrm{Tr}(t^a t^b + t^b t^a)t^c, \qquad d^{abc}d^{abd} = \frac{N^2-4}{N}\,\delta^{cd}$$

(one proves this using the main Fiertz identity). There is also a "symmetric Fiertz identity"

$$d^{abc}\,(t^a)^\alpha_\beta\,\left(t^b\right)^\gamma_\delta = \frac{1}{2}\,(t^c)^\alpha_\delta\;\delta^\gamma_\beta + \frac{1}{2}\,(t^c)^\gamma_\beta\;\delta^\alpha_\delta - \frac{1}{N}\,(t^c)^\alpha_\beta\;\delta^\gamma_\delta - \frac{1}{N}\,(t^c)^\gamma_\delta\;\delta^\alpha_\beta$$

The essence of gauge invariance is the requirement that one should be able to perform $SU(N)$ rotations locally, i.e. the parameters of the $SU(N)$ matrices can depend on space-time. This causes a difficulty since any kinetic-energy term for fields has derivatives which differentiate the parameters of the $SU(N)$ rotation. To overcome the difficulty, one introduces the Yang–Mills covariant derivative,

$$\nabla_\mu \stackrel{d}{=} \partial_\mu\,\mathbf{1} - iA^a_\mu t^a,$$

where A^a_μ is called the YM field or YM potential or *connection*. It is similar to the electromagnetic 4-potential but has $(N^2-1)\cdot d$ components. Under gauge transformation it transforms according to

$$A_\mu = A^a_\mu\,t^a \to U\,A_\mu\,U^\dagger + iU\,\partial_\mu U^\dagger$$

leading to the transformation of the covariant derivative

$$\nabla_\mu \to U\,\nabla_\mu\,U^\dagger.$$

Consider the commutator of two ∇'s:

$$[\nabla_\mu \nabla_\nu] = -iF_{\mu\nu},$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu A_\nu] = F^a_{\mu\nu} t^a,$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}\,A^b_\mu\,A^c_\nu.$$

$F^a_{\mu\nu} = -F^a_{\nu\mu}$ is called the YM field strength or YM *curvature*; it has $(N^2-1)\cdot d(d-1)/2$ components, but not all of them are independent as they are expressed through $(N^2-1)\cdot d$ potentials A^a_μ. The fact that not all components of $F_{\mu\nu}$ are independent is expressed by the Bianchi identity (aka Jacobi identity) which follows from the definition,

$$[\nabla_\lambda[\nabla_\mu\nabla_\nu]] + [\nabla_\mu[\nabla_\nu\nabla_\lambda]] + [\nabla_\nu[\nabla_\lambda\nabla_\mu]] = -i\left([\nabla_\lambda F_{\mu\nu}] + [\nabla_\mu F_{\nu\lambda}] + [\nabla_\nu F_{\lambda\mu}]\right) \equiv 0$$

or, since it is antisymmetric in the permutations of $\lambda, \mu\nu$,

$$\epsilon^{\kappa\lambda\mu\nu}\nabla_\lambda F_{\mu\nu} \equiv 0.$$

I stress that it is "identically zero": Bianchi identity does not restrict the fields as equations of motion do, but are satisfied by any field, just from the definition of the field strength $F_{\mu\nu} = i[\nabla_\mu \nabla\nu]$.

In modern (mathematized) language

$$\begin{aligned} A_\mu &= \text{“connection''} \\ F_{\mu\nu} &= \text{“curvature''} \end{aligned}$$

We can now construct a perfect invariant under local $SU(N)$ rotations,

$$\text{Tr}\, F_{\mu\nu} F_{\mu\nu} = \tfrac{1}{2} F^a_{\mu\nu} F^a_{\mu\nu}.$$

This is the generalization of the EM lagrangian, $E^2 - B^2$, to the non-Abelian gauge group. It contains terms quadratic, cubic and quartic in the YM field. Thus, YM theory is self-interacting even without fermions, as it follows from the requirement of gauge invariance.

The Euclidean YM partition function is

$$\mathcal{Z} = \int DA_\mu \, \exp\left(-\frac{1}{2g^2} \int d^d x \, \text{Tr}\, F_{\mu\nu} F_{\mu\nu}\right)$$

where g^2 is the gauge coupling constant; $[g^2] = [m]^{4-d}$. In $4d$ it is dimensionless.

The very simply-written YM action in fact encodes enormously rich dynamics. First, it is believed that the theory leads to the confinement of color – a unique phenomenon having no analogs in the history of physics. Second, it is believed that the originally massless YM fields A_μ disappear from the physical spectrum. Instead, gauge-invariant or 'colorless' bound states must appear, which have a nonzero mass. Third, the whole rich realm of strong (or nuclear) interactions, from π's to ^{238}U is, in principle, deducible from Quantum Chromodynamics (=QCD) being nothing but YM theory based on the $SU(3)$ gauge group.

($\$1 \cdot 10^6$ prize..)