2-dimensional $CP(N)$ model.

Asymptotic freedom. Restoration of symmetry. Instantons

$CP(N-1)$ model in $d=1+1$ (from 'Complex Projective')

The main objects of the CP^{N-1} model are N complex fields $v_A(x),\ A=1\ldots N$ or

$$u_A = \frac{v_A}{|v|}, \qquad |v| = \sqrt{\sum_{A=1}^{N} |v_A|^2}, \qquad \sum_A |u_A|^2 = 1, \qquad 2N-1 \ \text{d.o.f.}'\text{s.}$$

One can formally introduce a vector potential A_μ and a covariant derivative ∇_μ:

$$\begin{aligned} A_\mu &= \frac{i}{2}(u_A \partial_\mu u_A^* - u_A^* \partial_\mu u_A), \\ \nabla_\mu &= \partial_\mu - iA_\mu, \qquad (\mu = 1, 2). \end{aligned}$$

The theory is determined by the action

$$S = \int d^2x \sum_{A=1}^{N_C} |\nabla_\mu u_A|^2$$

and the partition function is

$$\begin{aligned} \mathcal{Z} &= \int Du_A(x) Du_A^*(x) DA_\mu(x)\, \delta(|u|^2 - 1) \\ &\cdot \exp\left(-\frac{1}{g^2}\int d^2x |\nabla_\mu u_A|^2\right) \end{aligned}$$

This formulates a nonlinear theory of self-interacting fields, where the nonlinearity is forced by the condition $|u| = 1$.

Notice that there is no 'photon energy' $F_{\mu\nu}^2$. Photon field A_μ is, thus, auxiliary field, and one may wish to integrate it out.

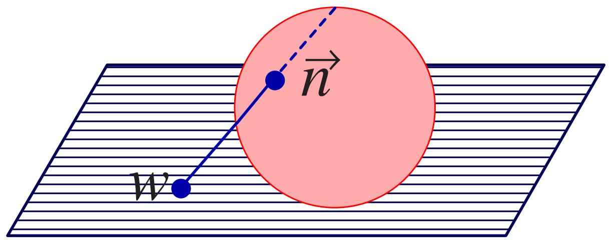
$$|\nabla_\mu u_A|^2 = \partial_\mu u^* \partial_\mu u + iA_\mu(u^*\,\partial_\mu u - \partial_\mu u^*\,u) + A_\mu^2.$$

The integral over A_μ is Gaussian, and one can integrate it out by the usual shift of integration variables. One gets

$$S = \int d^2x \, \left(\partial_\mu u_A^* \partial_\mu u_A + (u_A^* \partial_\mu u_A)(u_B^* \partial_\mu u_B)\right).$$

At $N = 2$ the $CP(1)$ model is actually identical to the $O(3)$ model considered above; let us show it. We return to the $O(3)$ model and make the stereographic parametrization of the unit $3d$ vector $\mathbf{n}$:

$$\begin{aligned} n_1 &= \frac{2w_1}{1+|w|^2}, \qquad w = w_1 + i w_2, \\ n_2 &= \frac{2w_2}{1+|w|^2}, \qquad w^* = w_1 - i w_2, \\ n_3 &= \frac{1-|w|^2}{1+|w|^2}, \qquad \mathbf{n}^2 = 1. \end{aligned}$$

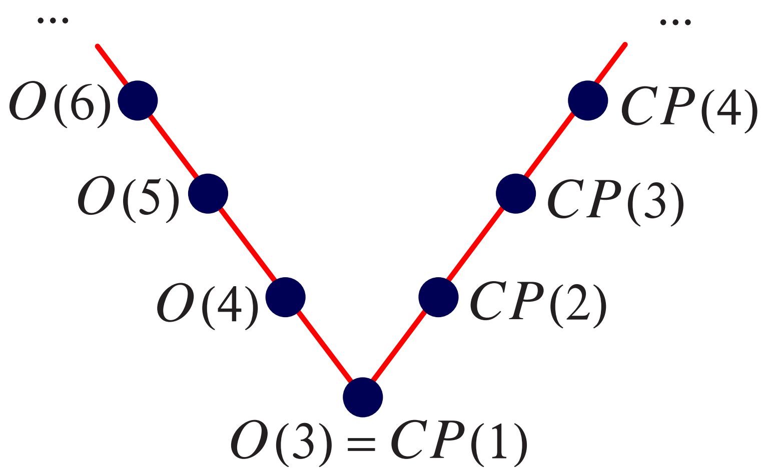


$$(\partial_\mu \mathbf{n})^2 = \frac{\partial_\mu w^* \, \partial_\mu w}{(1+|w|^2)^2}.$$

Equivalence between $O(3)$ and $CP(1)$ models is obtained by identifying the fields according to

$$w(x) = \frac{u_1(x)}{u_2(x)}. \qquad \text{Please check it!}$$

Thus, $O(3)$ model $=$ $CP(1)$ model but there are no further coincidence: $O(N)$ and $CP(N-1)$ models are two different branches stemming from one common root, $CP(1)$.

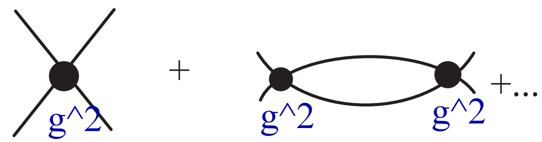


Asymptotic freedom of $CP(N-1)$

Similar to $O(N)$ model the $CP(N-1)$ model is asymptotically free, meaning that the dimensionless coupling constant $g^2(\mu)$ decreases logarithmically when the UV cutoff momentum $|k|_{\max} = \mu$ increases,

$$\frac{2\pi}{g^2(\mu)} = N \, \ln \frac{\mu}{\Lambda} + \ldots$$

One can see it in a variety of ways:



1. From Feynman graphs in perturbation theory:
2. From Friedan's method in curved background space
3. From $1/N$ expansion [in a moment..]
4. From instantons

Asymptotic freedom means that long wavelength quantum fluctuations interact strongly, and something clever should be done to deal with it.

$1/N$ expansion. Appearance of the photon

Let us get rid of the restriction $|u| = 1$ by introducing an auxiliary integration over a Lagrange multiplier field, like we have done it in the $O(N)$ model. Similar to that model, we shall see that λ develops a nonzero vacuum expectation value. However in $CP(N-1)$ there will be an additional interesting nonperturbative phenomenon.

We write down the partition function as

$$\mathcal{Z} = \int D\lambda\, DA_\mu\, Du_A^*\, Du_A \,\exp\left[-\frac{1}{g^2}\int d^2x \left((|\nabla_\mu u|^2 + \lambda |u|^2) - \lambda\right)\right].$$

Integration over $\lambda(x)$ along the imaginary axis at all space points x provides the needed $\prod_x \delta(|u(x)|^2 - 1)$. However, we wish to integrate over N complex fields $u_A,\ A = 1 ... N$ first. It is now an unrestricted Gaussian integral:

$$
\begin{aligned}
&\int Du_A^*\, Du_A\, \exp\left[-\frac{1}{g^2}\int d^2x\left(|\nabla_\mu u|^2 + \lambda|u|^2\right)\right] \\
&= \quad \exp\left[-N\,\mathrm{Sp}\ (\ln\ \mathcal{D} - \ln\ \mathcal{D}_0)\right],
\end{aligned}
$$

$$
\mathcal{D} \quad = \quad -\nabla_\mu^2 + \lambda, \qquad \mathcal{D}_0 = -\partial_\mu^2.
$$

To start with, we temporarily switch off the photon field A_μ ; then $\nabla_\mu^2 \longrightarrow \partial^2$, and the functional trace is the same as in the $O(N)$ model,

$$
\mathrm{Sp}\ (\ln\ \mathcal{D} - \ln\ \mathcal{D}_0) = \int d^2x \int \frac{d^2k}{(2\pi)^2}\ \ln\frac{k^2+\lambda}{k^2}.
$$

The vacuum expectation value (=v.e.v.) of the Lagrange multiplier field is found from the saddle point

$$\frac{\partial}{\partial\lambda}\left(\frac{\lambda}{g^2} - N\int\frac{d^2k}{(2\pi)^2}\ln\frac{k^2+\lambda}{k^2}\right) = 0,$$

leading to the nonzero

$$<\lambda>=\mu^2\exp\left(-\frac{4\pi}{Ng^2(\mu)}\right), \qquad \mu = \mathrm{UV\ cutoff}.$$

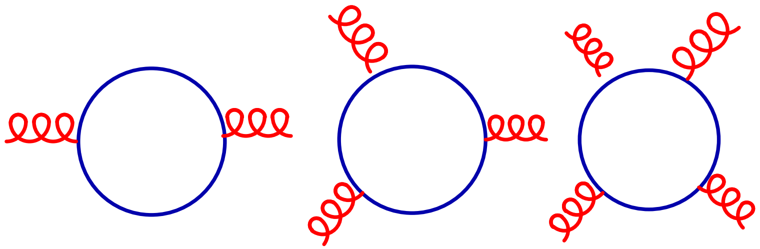
Notice that $<\lambda>$ is non-analytic in the coupling constant, therefore this result cannot be obtained in any finite order of the perturbation theory. The appearance of nonzero $<\lambda>$ means that $2N-1$ would-be Goldstone modes of the perturbation theory obtain masses: there are now $2N$ massive particles with the mass $m=\sqrt{<\lambda>}$.

Now let us recall the photon field A_μ. The evaluation of field-dependent functional determinants is an interesting (and tricky) business; it will be considered later. Now we notice that the operator $-\nabla^2+\lambda$ is gauge invariant, therefore, the functional trace can depend only on gauge-invariant combinations of A_μ. The first invariant is

$$F_{\mu\nu}^2, \qquad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu,$$

having the dimension of $1/\mathrm{cm}^4$. It appears in the expansion of the functional trace:

$$\exp\left(-N\,\mathrm{Sp}\;\ln\left(-\nabla^2+\lambda\right)\right)\to\exp\left(-N\int d^2x\,\frac{F_{\mu\nu}^2}{\lambda}\right).$$



Proof that $\det(-\nabla^2 + \lambda)$ is gauge invariant

$$\det(-\nabla^2 + \lambda) = \det\left(e^{-i\alpha(x)}(-\nabla^2 + \lambda)e^{i\alpha(x)}\right)$$

$$e^{-i\alpha(x)}\nabla_\mu e^{i\alpha(x)} = e^{-i\alpha(x)}\left(\partial_\mu - iA_\mu(x)\right)e^{i\alpha(x)} = \partial_\mu - i\left(A_\mu(x) - \partial_\mu\alpha(x)\right)$$

hence

$$\det\left(-\nabla^2(A_\mu) + \lambda\right) = \det\left(-\nabla^2(A_\mu - \partial_\mu\alpha) + \lambda\right)$$

meaning that this functional determinant can depend only on gauge-invariant combinations of the A_μ field, like $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$!

Thus, the fate of the $CP(N-1)$ model is the following. Like the $O(N)$ model, it is a field-theoretic model describing interacting $2N-1$ massless Goldstone modes, with additional gauge invariance but photons are not dynamical degrees of freedom and can be integrated out. It is asymptotically free, meaning that fluctuations with long wavelengths interact more and more strongly. It indicates that something dramatic must happen, but one cannot see it in perturbation theory.

In fact, the would-be Goldstone modes acquire a mass gap, and the photon field becomes dynamic. The true (nonperturbative) content of the model is N massive charged particles u_A, u_A^* which interact via photon exchange. In $2d$ the Coulomb interaction is linearly rising in the separation between charges.

In addition, this model has interesting classical solutions, – instantons.

Further reading on $CP(N-1)$ (original papers):

1. A. D'Adda, P. Di Vecchia and M. Lüscher, *Nucl. Phys.* B146 (1978) 63

2. E. Witten, *Nucl. Phys.* B149 (1979) 285

Instantons in $2d$ models

Below, we consider classical solutions in the $CP(N-1)$ model. Reminder: $CP(1) = O(3)$, the Heisenberg spin model. In this case, the classical solution (the instanton) has the meaning of a domain wall. However, it will be instructive to consider a more general case of $CP(N-1)$. The partition function is

$$\begin{aligned}\mathcal{Z} &= \int Du_A(x)Du_A^*(x)DA_\mu(x)\,\delta(|u|^2-1)\\ &\cdot \quad \exp\left(-\frac{1}{g^2}\int d^2x|\nabla_\mu u_A|^2\right), \qquad u_A(x)=\frac{v_A}{|v|},\end{aligned}$$

where $v_A(x)$ are N complex fields. We remind that the gauge potential A_μ is not a dynamical dof: it is equal to $\frac{i}{2}(u_A\partial_\mu u_A^* - u_A^*\partial_\mu u_A)$. The corresponding field strength is

$$\begin{aligned}F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu = i(\partial_\mu u_A^*\partial_\nu u_A - \partial_\nu u_A^*\partial_\mu u_A)\\ &= \epsilon_{\mu\nu}\,q(x)\end{aligned}$$

where $q(x)$ is called the topological charge density.

It will be useful to introduce a $N \times N$ projector P and a unitary matrix G such that

$$P_{AB} = u_A u_B^*, \quad G = 1 - 2P, \quad P^2 = P = P^\dagger, \quad G^2 = 1.$$

The action can be rewritten as

$$S = \int d^2x \sum_{A=1}^{N_C} |\nabla_\mu u_A|^2 = \frac{1}{8} \int d^2x \, \mathrm{Tr} \, \partial_\mu G \partial_\mu G.$$

There is a topological charge in the theory defined as

$$\begin{aligned} Q &= \frac{1}{2\pi} \oint dx_\mu A_\mu = \frac{1}{4\pi} \int d^2x \, \epsilon_{\mu\nu} F_{\mu\nu} = \frac{1}{4\pi} \int d^2x \, q(x) \\ &= \frac{i}{16\pi} \int d^2x \, \epsilon_{\mu\nu} \, \mathrm{Tr} \, G \partial_\mu G \partial_\nu G. \end{aligned}$$

Because of its unitarity G has the property $G\partial G = -\partial G G$ which can be used to show

$$0 \le \int d^2x \mathrm{Tr} \, [(\partial_\mu G \pm i\epsilon_{\mu\nu} G \partial_\nu G) \, (\partial_\mu G \pm i\epsilon_{\mu\nu} G \partial_\nu G)] = 16(S \mp 2\pi Q) \quad .$$

This tells us that the minimal action for a given topological charge of a field configuration, i.e. the solution of the equation of motion, is obtained if the field satisfies the self-duality equation,

$$\partial_\mu G \pm i\epsilon_{\mu\nu} G \partial_\nu G = 0 \quad \Leftrightarrow \quad \partial_\mu v_A \mp i\epsilon_{\mu\nu} \partial_\nu v_A = 0.$$

Introducing the complex coordinates in the 2d plane, $z = x_1 + ix_2$, $z^* = x_1 - ix_2$, and the complex derivative $\partial_z = (\partial_x - i\partial_y)/2$, the last Eqn can be written as the Cauchy–Riemann conditions:

$$\partial_z^* v_A = 0 \;\longleftrightarrow\; S = 2\pi Q$$

and

$$\partial_z v_A = 0 \;\longleftrightarrow\; S = -2\pi Q \ ,$$

Correspondingly, one finds two types of solutions: the analytic solution is called an instanton and the anti-analytic one is called the anti-instanton.

Why is the instanton a domain wall?

A general multi-instanton solution of the self-duality Eqn with topological charge $Q = N_+$ is a product of monomials in z:

$$v_A^{\rm inst} = c_A \prod_{i=1}^{N_+} (z - a_{Ai}), \quad A = 1, ..., N.$$

Similarly, a general multi anti-instanton solution with topological charge $Q = -N_-$ is a product of monomials in the complex conjugate variable z^*:

$$v_A^{anti} = c'_A \prod_{j=1}^{N_-} (z^* - b^*_{Aj}), \quad A = 1, ..., N_c.$$

For a general configuration with N_+ instantons and N_- anti-instantons we shall consider a product Ansatz:

$$v_A = \prod_{i=1}^{N_+}(z - a_{Ai}) \prod_{j=1}^{N_-}(z^* - b^*_{Aj}),$$

where a_{iA} and b_{jA} are fixed 2-dim points written as complex numbers. We shall call these points 'zindons'. There are, thus, N types ('colours') of the instanton zindons (a_A) and N types of anti-instanton zindons (b_A). For such a multi-instanton/ anti-instanton Ansatz the topological charge is found from the Cauchy theorem:

$$Q = \frac{1}{2\pi}\oint dx_\mu A_\mu = N_+ - N_- \,.$$

Let us take a single-instanton solution, $v_A(z) = c_A(z - a_A)$. Putting it into the action one gets

$$S = 2\pi Q = 2\int d^2x \frac{\rho^2}{[(x - x_0)^2 + \rho^2]^2} = 2\pi$$

where the instanton center x_0 is given by the 'center of masses' of zindons,

$$x_{0\mu} = \frac{\sum_{A=1}^{N_c} |c_A|^2 a_{A\mu}}{\sum_{A=1}^{N_c} |c_A|^2},$$

while the spread of the field called the instanton size ρ is given by the dispersion of zindons comprising the instanton,

$$\rho^2 = \frac{\sum_{A=1}^{N_c} |c_A|^2 |a_A - x_0|^2}{\sum_{A=1}^{N_c} |c_A|^2}.$$

All coordinates $x_0,\ a_A$ are understood as $2d$ vectors.

Let us consider the simplest instanton in $CP(1) = O(3)$ model. Recall

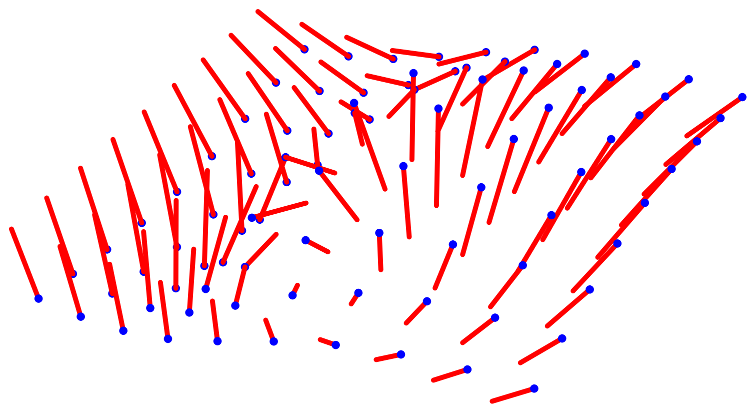
$$\begin{aligned} n_{1,2} &= \frac{2w_{1,2}}{1+|w|^2}, \quad w = w_1 + iw_2 = \frac{v_1}{v_2} = \frac{z-a_1}{z-a_2}, \\ n_3 &= \frac{1-|w|^2}{1+|w|^2}, \qquad \mathbf{n}^2 = 1. \end{aligned}$$

The instanton solution gives the following distribution of spin $\mathbf{n}(\mathrm{x})$ in space:

$$n_1 = \frac{(x-a_1)\cdot(x-a_2)}{(x-a_1)^2+(x-a_2)^2},$$

$$n_2 = \frac{\epsilon_{\mu\nu}(x-a_1)_\mu(x-a_2)_\nu}{(x-a_1)^2+(x-a_2)^2},$$

$$n_3 = \frac{(x-a_1)^2-(x-a_2)^2}{(x-a_1)^2+(x-a_2)^2}.$$



At $x=a_1$ $\mathbf{n}$ is sticking in the 3d direction, $n_3=1$, at $x=a_2$: $n_3=-1$.

This configuration can be understood in two different senses:

1) The system is a $1d$ chain of 'spins', and t is Euclidean time. Then the instanton is a time-dependent fluctuation of 'spins', a 'process', corresponding to a local minimum of the action.

2) The system is in $2d$. Then the above solution is a nonlinear static configuration of spins (a soliton), corresponding to a local minimum of the energy functional.

The difference, *inter alia*, is how one deals with quantum fluctuations about the classical solution. We shall treat it as an instanton.

Quantum fluctuations

General strategy in dealing with quantum oscillations:

1) Expand the general field about the classical solution,

$$v_A(x) = v_A^{\rm cl}(x, \xi_i) + y_A(x), \quad |y_A| \ll |v_A^{\rm cl}|,$$

where $\{\xi_i\}$ is a set of p collective coordinates characterizing a general solution; there will be p zero modes.

2) The expansion is not uniquely defined: there are p more dof's in the r.h.s. than in the l.h.s. One has to put p restrictions on the quantum field $y_A(x)$ such that it is 'orthogonal' to changing the values of collective coordinates. Introduce a unity into the functional measure,

$$
\begin{aligned}
1 \quad &= \quad \prod_{i=1}^{p} \int d\xi_i \int Dy_A(x)\, \delta\left(v_A(x) - v_A^{\mathrm{cl}}(x) - y_A(x)\right) \\
\cdot \quad & \delta\left(\int d^2x\, \psi_{Ai}(x,\xi) y_A(x)\right) \cdot \Phi[v_A(x)]
\end{aligned}
$$

where Φ is a functional making the r.h.s. identically unity. One finds (similarly to QM, see L-2):

$$
\Phi = \mathrm{det}_{\{ij\}} \int \psi_{Ai}(x,\xi)\, \frac{\partial v_A^{\mathrm{cl}}(x,\xi)}{\partial \xi_j}.
$$

3) The partition function of the theory in the semi-classical approximation becomes

$$\mathcal{Z} \;=\; \int D y_A(x) \prod_{i=1}^{p} \int d\xi_i \, \delta \left(\int dx \, \psi_{Ai}(x,\xi) \, y_A(x) \right)$$

$$\cdot \quad \mathrm{det}_{\{ij\}} \left(\int dx \, \psi_{Ai} \, \frac{\partial v_A^{\rm cl}}{\partial \xi_j} \right) \cdot \exp \left(-S \left[v_A^{\rm cl}(x,\xi) + y_A(t) \right] \right) \, .$$

This expression is in fact independent of the choice of the functions $\psi_{Ai}(t,\xi)$: the only restriction is that $\mathrm{det}_{\{ij\}} \neq 0$.

4) One expands the action about the classical solution. The linear term in y_A is absent (because the classical solution is found from the condition that the first variation of the action is zero). The quadratic term has the form

$$\int d^2x \, y_A(x) \, W_{AB}(x) \, y_B(x),$$

$$W_{AB} = \frac{\delta^2 S}{\delta v_A \delta v_B} = -\Delta\, \delta_{AB} + a_{AB\,\mu}(x,\xi)\,\frac{\partial}{\partial x_\mu} + b_{AB}(x,\xi).$$

5) One has to find the determinant of this second-order diff operator, which is the product of all its eigenvalues:

$$y(x) = \sum_n c_n\, y_n(x),$$

$$W y_n(x) = \lambda_n\, y_n(x), \qquad \int d^2x\, y_m\, y_n = \delta_{mn},$$

$$\int d^2x\, y\, W\, y = \sum_n c_n^2\, \lambda_n,$$

$$\int Dy(x) = \prod_n \int \frac{dc_n}{\sqrt{2\pi}} \qquad \Rightarrow \qquad (\det W)^{-\frac{1}{2}} = \left(\prod_n \lambda_n\right)^{-\frac{1}{2}},$$

6) Among eigenvalues λ_n there are p eigenvalues that are exactly zero $\Rightarrow$ zero modes: The saddle point has p flat directions! However, integration over those directions is trivial because of the p δ-functions:

$$\prod_{i=1}^{p}\int\frac{dc_i}{\sqrt{2\pi}}\,\delta\left(\int d^2x\,\psi_{Ai}(x,\xi)y_A(x)\right)_{y=c_1y_1+\ldots+c_py_p}$$

$$=\quad\left(\mathrm{det}_{\{ij\}}\int d^2x\,\psi_i\,y_j\right)^{-1},$$

which should be combined with $\det\int\psi_i\,(\partial v/\partial\xi_j)$ to produce the determinant made of the zero-mode normalization integrals,

$$\prod_{i=1}^{p}\int\frac{dc_i}{\sqrt{2\pi}}\to\left(\mathrm{det}_{\{ij\}}\int d^2x\,\frac{\partial v_A^{\mathrm{cl}}}{\partial\xi_i}\,\frac{\partial v_A^{\mathrm{cl}}}{\partial\xi_j}\right)^{\frac{1}{2}}\left(\frac{1}{2\pi}\right)^{\frac{p}{2}}.$$

Therefore, the partition function is (temporaly)

$$\mathcal{Z} = e^{-S}\left(\frac{1}{2\pi}\right)^{\frac{p}{2}}\prod_{i=1}^{p}\int d\xi_i\left(\det\int\frac{\partial v}{\partial\xi_i}\frac{\partial v}{\partial\xi_j}\right)^{\frac{1}{2}}(\mathrm{Det}'\,W)^{-\frac{1}{2}},$$

where Det′ is a product of nonzero eigenvalues of the quadratic form.

In case of the instanton of the $CP(N-1)$ model the collective coordinates ξ_i are $2d$ positions of the zeroes of $v_A(x)$. Their number (number of zero modes) is $p = 2NQ$, where Q is topological charge.

7) Det′ must be normalized (to the 'free' or perturbative determinant, – that was also the case in QM) and regularized at high eigenvalues; this is new!

Normalization to free – that is simple:

$$\mathrm{Det}\,W \to \frac{\mathrm{Det}\,W}{\mathrm{Det}\,W_0}, \quad W_0 = -\Delta\,\delta_{AB}.$$

Det W_0 is an (infinite) factor independent of the external field. UV regularization is tricky!

UV regularization:

Shift W by a large 'Pauli–Villars' mass μ^2:

$$\frac{\text{Det}\, W}{\text{Det}\, W_0} \quad \longrightarrow \quad \text{Det}'(W)^{\text{reg}}_{\text{norm}} = \frac{\text{Det}\, W}{\text{Det}\, W_0} \cdot \frac{\text{Det}(W_0 + \mu^2)}{\text{Det}(W + \mu^2)}$$

$$= \quad \prod_n \frac{\lambda_n\,(\lambda_{0n} + \mu^2)}{(\lambda_n + \mu^2)\,\lambda_{0n}}.$$

(the 'quadrupole formula'). Eigenvalues $\lambda_n \gg \mu^2$ are effectively cut out in this formula: only eigenvalues $\lambda_n \leq \mu^2$ are taken into account.

8) The regularization affects also the zero-mode contribution since p zero eigenvalues have to be shifted by μ^2. One gets an additional factor

$$\left(\frac{\mu^2}{g^2}\right)^{\frac{p}{2}}$$

9) Finally, assembling all factors together, we obtain the contribution of a classical solution to the partition function:

$$\begin{aligned}\mathcal{Z} &= e^{-S} \\ &\cdot \prod_{i=1}^{p} \int d\xi_i \left(\mathrm{det}_{\{ij\}} \int d^2x \, \frac{\partial v_A^{\rm cl}}{\partial \xi_i} \frac{\partial v_A^{\rm cl}}{\partial \xi_j} \right)^{\frac{1}{2}} \left(\frac{\mu^2}{2\pi g^2} \right)^{\frac{p}{2}} \\ &\cdot \mathrm{Det}'(W)^{\rm reg}_{\rm norm}.\end{aligned}$$

$CP(1)$, one instanton

$$v_1(x) = z - a_1, \qquad v_2(x) = z - a_2. \qquad \text{in } O(3) \text{ language} : w(x) = \frac{v_1(x)}{v_2(x)} = \frac{z - a_1}{z - a_2}.$$

The topological charge is $+1$. The action is

$$S = \frac{2}{g^2}\int d^2x \frac{\rho^2}{[(x-x_0)^2+\rho^2]^2} = \frac{2\pi}{g^2},$$

$$x_{0\mu} = \frac{(a_1+a_2)_\mu}{2}, \quad \rho_\mu = \frac{(a_1-a_2)_\mu}{2}.$$

There are $p=4$ collective coordinates $a_{1,2}$ characterizing one instanton and hence 4 exact zero modes of the quadratic form W for quantum fluctuations. The one-instanton contribution to the partition function is

$$\mathcal{Z}^{1\,\mathrm{inst}} = e^{-\beta}\int d^2a_1 d^2a_2\,(\beta\mu^2)^2\,(\text{det from zero modes})^{\frac{1}{2}}\cdot \mathrm{Det}'(W)^{\mathrm{reg}}_{\mathrm{norm}}$$

$$\beta \stackrel{d}{=} \frac{2\pi}{g^2(\mu)}$$

where $g^2(\mu)$ is the 'bare' coupling constant defined at some normalization momentum scale μ. The determinant made of the zero modes $\frac{\partial v_A^{\rm cl}}{\partial a_{1,2}}$ is dimensionless number of the order of unity. The functional determinant from nonzero modes is also dimensionless and can depend only on $\mu\rho$ where ρ is the size of the instanton configuration. The dependence on $\mu\rho$ can be found without calculations – from the known renormalization properties of the theory: $\mathcal{Z}$ t depend on the (arbitrary) UV cutoff μ. From L-3 we know that

$$\Lambda^2 = \mu^2 \exp\left(-\frac{2\pi}{g^2(\mu)}\right) = \mu^2 e^{-\beta}$$

is in fact independent of the UV cutoff μ. Hence $\mathrm{Det}'(W)^{\rm reg}_{\rm norm} = \mathrm{const.}\,(\mu\rho)^{-2}$.

$$\begin{aligned}\mathcal{Z}^{1\ \rm inst} &= \int d^2a_1 d^2a_2 \frac{\Lambda^2}{|a_1-a_2|^2}\,\mathrm{const.}\\ &= \int d^2x_0 \frac{d\rho}{\rho^3}(\Lambda\rho)^2\,\mathrm{const.}, \qquad \Lambda = \mu\,\exp\left(-\frac{\pi}{g^2(\mu)}\right)\end{aligned}$$

Instantons wish to swell, as $\int d\rho/\rho$ is logarithmically divergent!

$CP(1)$, many instantons

One has $p = 2 \cdot 2 \cdot N_+$ collective coordinates (N_+ is the number of instantons, here equal to the topological charge of the configuration) $a_{Ai},\ A = 1, 2;\ i = 1 ... N_+$. The action is $S = 2\pi N_+/g^2$, hence, from renormalizability the multi-instanton weight is proportional to Λ^{2N_+}. The dependence on zindon coordinates a_{Ai} was found in Fateev, Frolov and Schwarz, *Nucl. Phys.* B154 (1979) 1:

$$\mathcal{Z}^{\mathrm{N}_+\,\mathrm{inst}} = \int d^2 a_{Ai}\, \Lambda^{2N_+} \cdot \frac{\prod_{i<j} |a_{1i} - a_{1j}|^2 \prod_{i<j} |a_{2i} - a_{2j}|^2}{\prod_{i,j} |a_{1i} - a_{2j}|^2}.$$

This is the partition function of the $2d$ Coulomb gas : same-kind zindons repulse each other while opposite-kind attract each other.

$$\frac{1}{|a_1 - a_2|^2} = \exp\left(-\ln|a_1 - a_2|^2\right), \text{ etc.}$$

$CP(N-1)$, many instantons

There are $p = 2 \cdot N \cdot N_+$ collective coordinates: the positions of N_+ zindons of N different 'colours', a_{Ai}, $A = 1...N$, $i = 1...N_+$. The multi-instanton weight has been found in Fateev, Frolov and Schwarz, *Sov. J. Nucl. Phys.* 30 (1979) 590 and Berg and Lüscher, *Comm. Math. Phys.* 69 (1979) 57. It is a multi-component 'Coulomb gas': zindons of different 'colour' attract each other while those of the same colour repulse each other.

Actually, anti-instantons [a reminder: anti-instanton solutions are given by a product of anti-analytic monomials] are as good as instantons, and one has to mix them together with instantons.

Zindons and anti-zindons are attractive if they are of the same colour and repulsive otherwise.

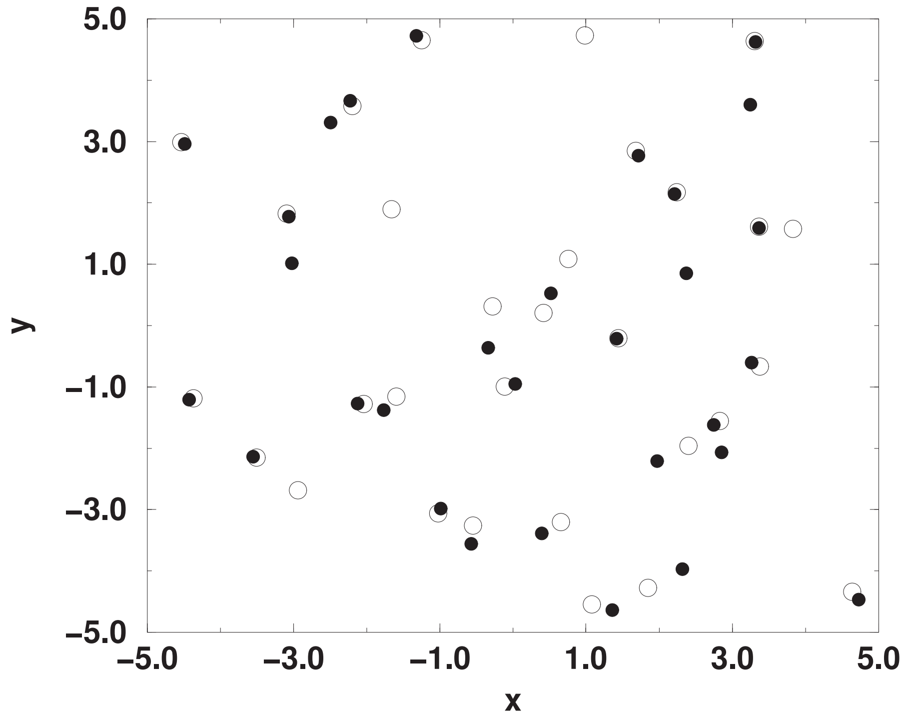
The resulting statistical mechanics of many I's and $\bar{I}$'s was studied by D.D. and M. Maul. Although I's and $\bar{I}$'s are allowed to overlap, dynamically they prefer not to do so.

The fate of the Coulomb gas: the Debye screening in plasma: correlation functions decay **exponentially** (and not power-like) at large distances! The correlation length is $\sim 1/\Lambda$ (there is no other dimensional parameter in the theory), owing to the transmutation of dimensions.

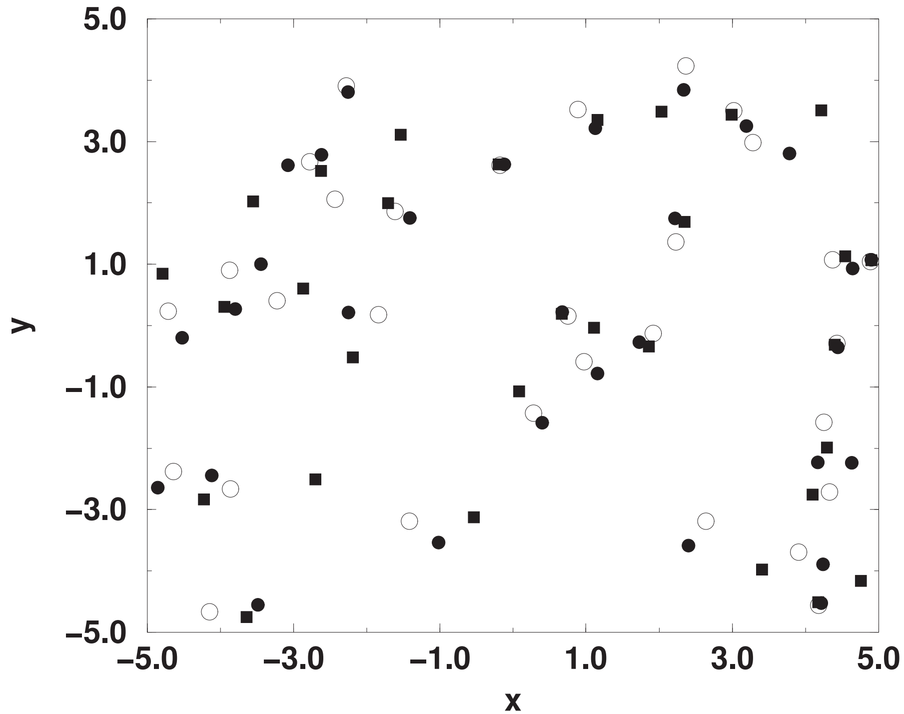
Instantons disorder fields (spins) to such extent that correlation functions decay much faster than without instantons.

The transmutation of dimensions arises from integration over high-frequency small quantum fluctuations about classical solutions (instantons).

At large N we have derived the spontaneous mass generation in a more simple way, not referring to instantons. Instantons provide a microscopic insight into how the mass is generated in the theory, at any N.



Distribution of ‘zindons’ in $O(3) = CP(1)$. Open and closed circles (making up an instanton) are strongly correlated. From: D. Diakonov and M. Maul, *Nucl. Phys.* B571 (2000) 91.



Distribution of ‘zindons’ in $CP(2)$. Open circles, black circles and black squares (making up an instanton) are strongly correlated. From: D. Diakonov and M. Maul, *Nucl. Phys.* B571 (2000) 91.

Summary of $CP(N-1)$ model

- The model is asymptotically free: the coupling decreases at small distances and blows up at large distances

- The model is solvable in the limit $N \to \infty$: a mass gap is spontaneously generated from the v.e.v. of the Lagrange multiplier field, via the transmutation of dimensions

- The model possesses classical solutions – instantons and anti-instantons (analytical and anti-analytical functions) having the meaning of spin domain walls

- The ensemble of instantons and anti-instantons resembles a multi-component Coulomb plasma. The correlation functions decay exponentially because of the Debye screening in plasma

- Instantons lead to an additional disordering of the fields, and that can be viewed as the microscopic explanation of the appearance of the mass gap, at any N

Problems

1. Show that the instanton solution

$$v_A(x) = z - a_A, \qquad z = x + iy, \qquad A = 1 \ldots N,$$

corresponds to the action density

$$\frac{\rho^2}{((\mathbf{x} - \mathbf{x_0})^2 + \rho^2)^2}$$

where

$$\begin{aligned} \mathbf{x_0} &= \frac{1}{N} \sum_{A=1}^{N} \mathbf{a_A}, \\ \rho^2 &= \frac{1}{N} \sum_{A=1}^{N} |\mathbf{a_A} - \mathbf{x_0}|^2. \end{aligned}$$

2. Show that the instanton action is

$$\begin{aligned} S &= \frac{1}{8}\int d^2x\,\mathrm{Tr}\;\partial_\mu G\partial_\mu G = 2\pi, \\ G_{AB} &= \delta_{AB} - 2\,u_A u_B^*, \qquad u_A = \frac{v_A}{|v|}, \end{aligned}$$

and the topological charge is

$$\frac{i}{16\pi}\int d^2x\,\epsilon_{\mu\nu}\,\mathrm{Tr}\;G\partial_\mu G\partial_\nu G = 1.$$