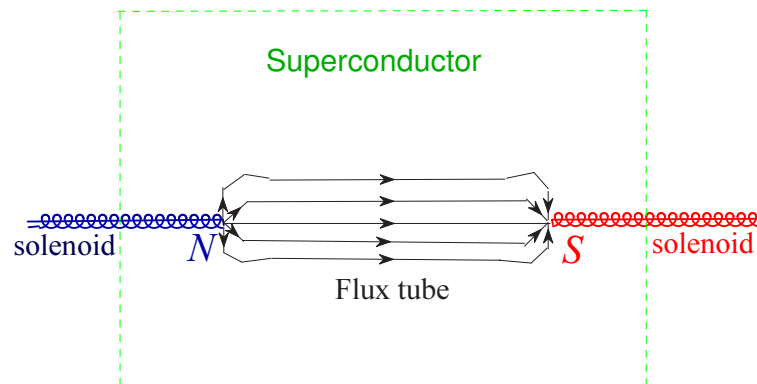


Mandelstam – Polyakov – 't Hooft mechanism of confinement or dual Meissner effect

The usual Meissner effect: magnetic field cannot penetrate into the superconductor (except by burning out a narrow tube where superconductivity is destroyed = Abrikosov vortex).

Two infinitely thin and long solenoids are, at their endpoints, sources of the Coulomb-like magnetic field.



Energy of the magnetic 'monopole-antimonopole' pair = $E_{\perp} \cdot L \implies$ linear potential energy between monopoles.

Dual Meissner effect:

- condensation of magnetic monopoles
- quarks are sources of (colored) electric field
- Along the tube connecting quarks the magnetic condensate is destroyed
- electric field squeezed inside the tube
= Abrikosov–Nielsen–Olesen vortex

Estimate of the string tension

Landau–Ginsburg effective theory of superconductivity:

$$\mathcal{E} = \int d^3r \left[\frac{\mathbf{B}^2}{2} + |(\partial_i - ieA_i)\phi|^2 + \lambda^2(\phi^2 - v^2)^2 \right],$$
$$m_W = ev, \quad m_H = \lambda v, \quad \mathbf{B} = \text{curl } \mathbf{A}.$$

Dimensionless quantities: $\phi' = \frac{\phi}{v}$, $A'_i = \frac{A_i}{m_W}$,

$$x = rm_W, \quad \kappa = \frac{\lambda}{e}.$$

$$\mathcal{E} = Lv^2 \int d^2x \left[\frac{\mathbf{B}'^2}{2} + |(\partial_i - ieA'_i)\phi'|^2 + \kappa^2(\phi'^2 - 1)^2 \right],$$

$$\text{vortex transverse size } \rho_0 \sim \frac{1}{\sqrt{\kappa}}.$$

$$\text{String tension} = \sigma = \text{energy} / \text{length of the tube} = v^2 \left[O(1) + O\left(\frac{m_H}{m_W}\right) \right].$$

London's limit: $m_H \rightarrow \infty \implies$ infinite-energy vortex

Bogomolny–Prasad–Sommerfeld limit: $m_H \rightarrow 0$.

type-I superconductor: ($m_H > m_W$): no vortices

type-II superconductor: ($m_H < m_W$): **yes**

One needs an analog of type II superconductor with **magnetic monopoles condensed**.

Polyakov's realization of confinement

$d = 2 + 1$ Georgi–Glashow model: Yang–Mills SU(2) fields interacting with the Higgs field in the triplet representation:

$$S = \int d^3x \left\{ \frac{(F_{ij}^a)^2}{4g^2} + \frac{1}{2} (D_i^{ab} \phi^b)^2 + \lambda^2 [(\phi^a)^2 - v^2]^2 \right\}.$$

't Hooft–Polyakov monopole is a local minimum of this action:

$$\phi^a = \mp n^a v \Phi(r), \quad n^a = \frac{x^a}{r}, \quad \Phi(r) \rightarrow \begin{cases} 0, & r \rightarrow 0 \\ 1, & r \rightarrow \infty \end{cases}$$

$$A_i^a = \epsilon_{aij} n_j \frac{1 - R(r)}{r}, \quad R(r) \rightarrow \begin{cases} 1, & r \rightarrow 0 \\ 0, & r \rightarrow \infty \end{cases}$$

Magnetic field strength, $B_i \stackrel{r \rightarrow \infty}{\sim} \frac{n_i}{r^2}$, is that of the magnetic monopole !

$$\text{Action} = \frac{v}{g} w \left(\frac{m_H}{m_W} \right) \gg 1, \quad w(0) = 4\pi \text{ (in the BPS limit)}$$

To add up hedgehogs, one has first to **comb their hair!**

'Stringy' or singular gauge

The theory is invariant under gauge transformations:

$$\begin{aligned} \phi &= \phi^a \frac{\tau_a}{2} \rightarrow S(x) \phi S^\dagger(x), \\ A_i &= A_i^a \frac{\tau_a}{2} \rightarrow S(x) A_i S^\dagger(x) + i S \partial_i S^\dagger. \end{aligned}$$

Choose the unitary gauge-transformation matrix $S(x)$ such that

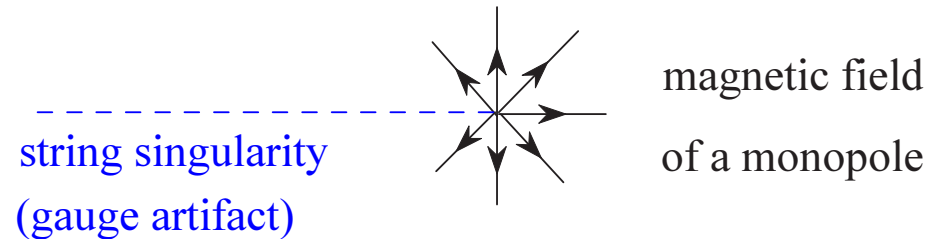
$$S(\mathbf{n} \cdot \boldsymbol{\tau}) S^\dagger = \tau_3 \implies S(\theta, \phi) = e^{-i\frac{\phi}{2}\tau_3} e^{i\frac{\theta}{2}\tau_2} e^{i\frac{\phi}{2}\tau_3}.$$

$$\phi' = S \phi S^\dagger = \frac{\tau_3}{2} v \Phi(r) \rightarrow \frac{\tau_3}{2} v,$$

$$A'_i = \begin{cases} A'_r = & 0 \\ A'_\theta = & \text{only inside core} \\ A'_\phi = & \text{inside core } \mp \frac{\tau_3}{2r} \frac{1-\cos\theta}{\sin\theta}. \end{cases}$$

Magnetic field strength outside the monopole core at $r \gg 1/m_W$:

$$B'_r = (\text{curl } A')_r = \frac{1}{r \sin\theta} \frac{\partial}{\partial\theta} (\sin\theta A'_\phi) = \pm \frac{\tau_3}{2r^2}.$$



Monopole interactions

One can add up (anti)monopoles only in the singular ('stringy') gauge where the Higgs field $\phi^a \xrightarrow{r \rightarrow \infty} \delta^{a3} v$.

The interaction of two (anti)monopoles is Coulomb-like at large separations:

$$U_{\text{int}} = \frac{4\pi}{g^2} \frac{1}{r_{12}} \left(\pm 1 - e^{-r_{12} m_H} \right).$$

Two gluons (out of three) $W^\pm = \frac{A^1 \pm iA^2}{\sqrt{2}}$ have large masses $m_W = gv$ and decouple. The third gluon (the 'photon') is massless, but there are monopoles around.

Monopole ensemble

Monopole 'weight' or 'fugacity'

$$\zeta = (\text{pre-exponent}) \cdot \exp(-\text{Action}) = \text{controllably small.}$$

At small momenta, the theory becomes the theory of plasma of magnetic charges of

opposite sign. The grand canonical partition function is

$$\mathcal{Z} = \sum_{N_+, N_-} \frac{\zeta^{N_+} \zeta^{N_-}}{N_+! N_-!} \prod_m \int d^3 z_m \exp \left(-\frac{4\pi}{g^2} \sum_{m < n} \frac{q_m q_n}{|z_m - z_n|} \right).$$

“Bosonization trick”

Monopole charge density

$$\rho(x) = \sum q_m \delta(x - z_m)$$

Consider a functional integral

$$\begin{aligned} & \int Dw(x) \exp \left(- \int d^3 x \left[\frac{1}{2} (\partial_i w)^2 - i \frac{4\pi}{g} w \rho \right] \right) \\ &= \exp \frac{1}{2} \int d^3 x \left(\frac{4\pi}{g} \right)^2 \rho \frac{1}{\Delta} \rho \end{aligned}$$

$$\begin{aligned}
&= \exp -\frac{1}{2} \frac{4\pi}{g^2} \iint d^3x d^3y \rho(x) \frac{1}{|x-y|} \rho(y) \\
&= \exp -\frac{4\pi}{g^2} \sum_{m < n} \frac{q_m q_n}{|z_m - z_n|}.
\end{aligned}$$

Therefore, the plasma partition function can be written down as a functional integral:

$$\mathcal{Z} = \sum_{N_{\pm}} \frac{\zeta^{N_+} \zeta^{N_-}}{N_+! N_-!} \int D w e^{-\frac{1}{2} \int d^3x (\partial_i w)^2} \prod_m \int d^3 z_m e^{i \frac{4\pi}{g} \sum q_m w(z_m)}$$

Inside, is a series for the exponent:

$$\sum_N \frac{\zeta^N}{N!} \left(\int d^3 z e^{\pm i \frac{4\pi}{g} w(z)} \right)^N = \exp \left(\int d^3 z e^{\pm i \frac{4\pi}{g} w(z)} \right).$$

Hence

$$\mathcal{Z} = \int Dw \exp \left\{ - \int d^3x \left[\frac{1}{2} (\partial_i w)^2 - 2\zeta \cos \left(\frac{4\pi}{g} w \right) \right] \right\}.$$

$$-2\zeta \cos \left(\frac{4\pi}{g} w \right) \simeq -2\zeta + \frac{1}{2} \mu^2 w^2,$$

$$\mu^2 = 2\zeta \left(\frac{4\pi}{g} \right)^2 \quad \text{Debye mass!}$$

The plasma partition function is mathematically equivalent to the 'Sine-Gordon' field theory. The field w has the meaning of the dual potential: it gets a nonzero mass owing to the Debye screening in the monopole plasma.

$$\mathcal{Z} = \exp \left\{ 2\zeta V + \frac{1}{12\pi} \left[2\zeta \left(\frac{4\pi}{g} \right)^2 \right]^{\frac{3}{2}} V + \dots \right\}$$

Average monopole density:

$$\frac{\bar{N}}{V} = \frac{1}{V} \frac{\partial \ln \mathcal{Z}}{\partial \ln \zeta} = 2\zeta + \text{Coulomb corrections}$$

In the weak-coupling regime $g \ll v$ monopoles are heavy, their density is small, everything is under control, this theory becomes exact!

Other ways to present plasma

$$\mathcal{Z} = \sum_{N_{\pm}} \frac{\zeta^{N_+ + N_-}}{N_+! N_-!} \prod_m \int d^3 z_m e^{-\frac{4\pi}{g^2} \sum \frac{q_m q_n}{|z_m - z_n|}}$$

$$\times \int D\rho(x) \delta\left(\rho - \sum q_m \delta(x - z_m)\right)$$

$$\delta(\dots) = \int D\mu(x) \exp\left[i \int d^3 x \mu \rho - i \sum q_m \mu(z_m)\right]$$

$$= \int D\rho D\mu Dw \exp - \int d^3x \left[\frac{1}{2} (\partial_i w)^2 - 2\zeta \cos\left(\frac{4\pi}{g}w - \mu\right) - i\mu\rho \right].$$

New variables: $\frac{4\pi}{g}w - \mu = \psi$; $\frac{4\pi}{g}w + \mu = \phi \leftarrow$ integrate over

$$\mathcal{Z} = \int D\rho D\psi \exp \int d^3x \left[\frac{1}{2} \left(\frac{4\pi}{g}\right)^2 \rho \frac{1}{\Delta} \rho - i\psi\rho + 2\zeta \cos\psi \right]$$

$$= \int D\rho \exp \int d^3x \left[\frac{1}{2} \left(\frac{4\pi}{g}\right)^2 \rho \frac{1}{\Delta} \rho - V(\rho) \right]$$

$$V(\rho) = \rho \ln \left(\frac{\rho}{2\zeta} + \sqrt{1 + \frac{\rho^2}{4\zeta^2}} \right) + 2\zeta \sqrt{1 + \frac{\rho^2}{4\zeta^2}}$$

$$\xrightarrow{\rho \ll \zeta} -2\zeta + \frac{\rho^2}{4\zeta}$$

Unusual: $\text{div } B^3 = -4\pi\rho$, $\text{curl } B^3 = 0$ (!)

The 'kinetic energy' of monopole density is nothing but the magnetic energy:

$$-\int d^3x \frac{1}{2} \left(\frac{4\pi}{g}\right)^2 \rho \frac{1}{\Delta} \rho = \frac{1}{2g^2} \int d^3x B_i \partial_i \frac{1}{\Delta} \partial_j B_j = \int d^3x \frac{B_i B_i}{2g^2}.$$

Hence

$$\mathcal{Z} = \int DB_i \delta(\text{curl}B) D\psi \exp \int d^3x \left[-\frac{B^2}{2g^2} + i\frac{\text{div}B}{4\pi}\psi + 2\zeta \cos \psi \right].$$

Confinement (= Area law for large Wilson loops)

Wilson loop

$$W = \text{Tr } P \exp i \oint dx^i A_i^a \frac{\tau_a}{2}.$$

At large distances from monopoles only colour A_i^3 component survives \implies can use the Stokes theorem:

$$W = \text{Tr} \exp i \int d^2 S^i B_i^3 \frac{\tau_3}{2} = e^{\frac{i}{2}\Phi} + e^{-\frac{i}{2}\Phi},$$

where $\Phi = \int d^2 S^i B_i^3$ is the flux of the magnetic field created by monopoles and antimonopoles in the plasma, through the surface spanned over the Wilson loop.

Wilson loop **averaged** over the ensemble of monopoles: have to plunge the source term $\exp\left(\frac{i}{2} \int d^2 \vec{S} \cdot \vec{B}\right)$ into the partition function, and integrate over all possible magnetic fields.

For large loops (say, lying in the $z=0$ plane), one can use the saddle-point method and find the 'best' fields \vec{B} , ψ minimizing the energy together with the surface source:

$$B_i(x, y, z) = \delta_{iz} B(z), \quad \psi = \psi(z),$$

$$\frac{1}{g^2} B(z) + \frac{i}{4\pi} \frac{d\psi}{dz} = \frac{i}{2} \delta(z),$$

$$i \frac{dB}{dz} - 8\pi\zeta \sin \psi = 0,$$

for x, y inside the contour. The solution:

$$B = i \frac{g^2 \mu}{\pi} \frac{e^{-\mu|z|}}{1 + e^{-2\mu|z|}},$$

$$\psi = 4 \operatorname{sign}(z) \operatorname{atan} \left(e^{-\mu|z|} \right).$$

The solution corresponds to a purely imaginary double layer of monopoles around the surface.

String tension

$$\langle W \rangle = \exp(-\sigma \text{Area}),$$

$$\sigma = \frac{g^2 \mu}{2\pi^2} = \frac{2g}{\pi} \sqrt{2\zeta},$$

proportional to the *square root* of the mean monopole density $N/V = 2\zeta$.

No massless states are left in the theory. For example, consider the correlation function of the magnetic field (in momentum space):

$$\begin{aligned} & \langle B_i^3(p) B_j^3(-p) \rangle \\ &= g^2 \left(\delta_{ij} - \frac{p_i p_j}{p^2} \right) + (4\pi)^2 \frac{N}{V} \frac{p_i p_j}{p^4} - (4\pi)^2 \frac{N}{V} \frac{\mu^2}{p^2 + \mu^2} \frac{p_i p_j}{p^4} \\ &= g^2 \left(\delta_{ij} - \frac{p_i p_j}{p^2 + \mu^2} \right), \quad \text{Debye mass : } \mu^2 = 2\zeta \left(\frac{4\pi}{g} \right)^2. \end{aligned}$$

Best dreams fulfilled! [A. Polyakov, Nucl. Phys. B (1977)]. However,

i) $d = 2 + 1$ and ii) the gauge group $SU(2)$ is explicitly broken by the Higgs field down to $U(1)$ where the *dual* photon gets the mass.

More recent achievements:

- $\mathcal{N} = 2$, $d = 3 + 1$ supersymmetric theory, softly broken to $\mathcal{N} = 1$ supersymmetric theory. It is also shown to possess confinement and mass gap – more or less due to the same mechanism (of monopole condensation) [Seiberg and Witten (1994), Douglas and Shenker (1995)]
- pure Yang–Mills $d = 3 + 1$ non-supersymmetric theory [D.D. and Petrov (2007)] is also shown to possess confinement, at least at the semiclassical level!