Autumn 2008, Bochum

Instantons, solitons, monopoles ...

Lecture notes will be available at http://thd.pnpi.spb.ru/~diakonov

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Content of the course

1. Functional integrals in Quantum Mechanics, Statistical Mechanics and Field Theory. Imaginary time

2. Double well potential in Quantum Mechanics. Instantons = kinks

3. Quantum fluctuations about instantons

4. Instanton gas and level splitting

5. The simplest quantum field theory in $1+1$ dimensions: Heisenberg's model for ferromagnets and its generalizations. Two-dimensional quantum field models: O(N), CP(N), HP(N).

6. Asymptotic freedom in quantum field theory, dimensional transmutation

7. A generalization: Friedan's renormalization in curved space.

8. Nonperturbative $1/\mathrm{N}$ expansion. Spontaneous mass generation and restoration of symmetry in two dimensions.

9. Instantons in 2d models. Domain walls

10. Non-Abelian gauge theory (or Yang–Mills theory) in 2,3,4 dimensions

11. Yang–Mills theory on a lattice. Methods and results

12. Anatomy of the $SO(N)$ and $SU(N)$ groups: parametrization, metrics in the group space, and integrals over groups.

13. Yang–Mills theory at non-zero temperature

14. Classical solutions in Yang–Mills theory: monopoles and dyons

15. Classical solutions in Yang–Mills theory: instantons

16. Classical solutions in Yang–Mills theory: calorons with non-trivial holonomy.

17. Ensembles of dyons. Confinement of colour, and the confinement-deconfinement phase transition.

18. Quantum anomalies. Interpretation of anomalies

19. Quantum determinants: exact, approximate and numerical methods

20. Spontaneous chiral symmetry breaking in strong interactions

21. The Skyrmion. Quantum fluctuations about solitons.

Feynman's formulation of Quantum Mechanics:

Let us consider a non-relativistic particle with mass m in a one-dimensional potential well $V(x)$. Its Lagrangian is

$$L = \frac{m\dot{x}^2}{2} - V(x) \tag{1}$$

and its energy is $H = \frac{m\dot{x}^2}{2} + V(x)$. In Quantum mechanics one introduces the Hamiltonian

$$\mathcal{H} = -\frac{1}{2m}\frac{d^2}{dx^2} + V(x). \tag{2}$$

To find the (quantized) energy levels E_n and the stationary wave functions $\psi_n(x)$ one solves the Schrödinger eqn:

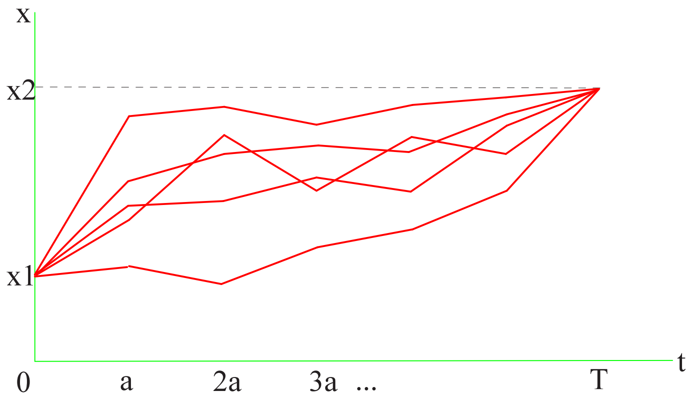
$$\mathcal{H}\psi_n(x) = E_n\,\psi_n(x) \tag{3}$$

The evolution operator is $e^{-i\mathcal{H}T}$ where T is time.

The probability amplitude that the particle goes from point x_1 to point x_2 during time T

$$A_{12} \quad = \quad \langle x_2 | e^{-\frac{iHT}{\hbar}} | x_1 \rangle = \sum_n \psi_n^*(x_2)\, e^{-iE_nT}\, \psi_n(x_1)$$

$$= \quad \int_{x(0)=x_1}^{x(T)=x_2} Dx(t)\ \exp \frac{i}{\hbar} \int_0^T dt \left[\frac{m\dot{x}^2(t)}{2} - V\left(x(t)\right) \right]$$



Path integral over trajectories
with x(0)=x1, x(T)=x2

Discretized action

$$S = \sum_n a \left[\frac{m}{2} \left(\frac{x(t_n) - x(t_{n-1})}{a} \right)^2 - V\left(x(t_n)\right) \right].$$

Path integral can be understood as the limit of an infinite number of ordinary integrations – over the intermediate points $x_1 \ldots x_N$:

$$A_{12} = \lim_{N\to\infty} \mathcal{N} \prod_{n=1}^{N} \int dx_n \, e^{\frac{i}{\hbar} S(x_1,\ldots,x_N)}.$$

If $S \gg \hbar$ quantum mechanics becomes classical: small fluctuations of trajectories lead to a large variation of the action, and phase factors for close trajectories $e^{iS/\hbar}$ annihilate each other. Only those trajectories contribute to the path integral, whose small variation doesn't change the action, $\delta S = 0$. But this condition is the Euler–Lagrange equation of motion:

$$-\frac{d}{dt}\frac{dL}{d\dot{x}} + \frac{dL}{dx} = 0\,!$$

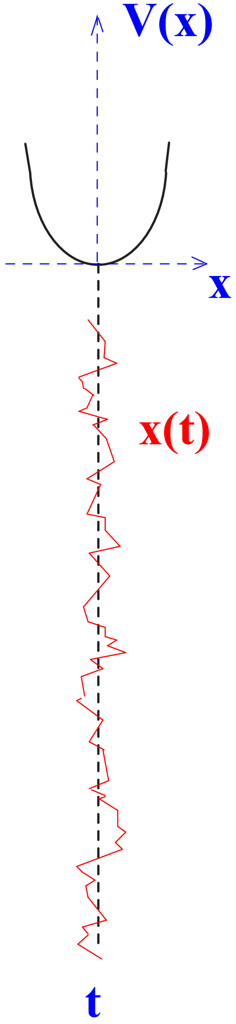
In classical theory one differentiates
In quantum theory one integrates

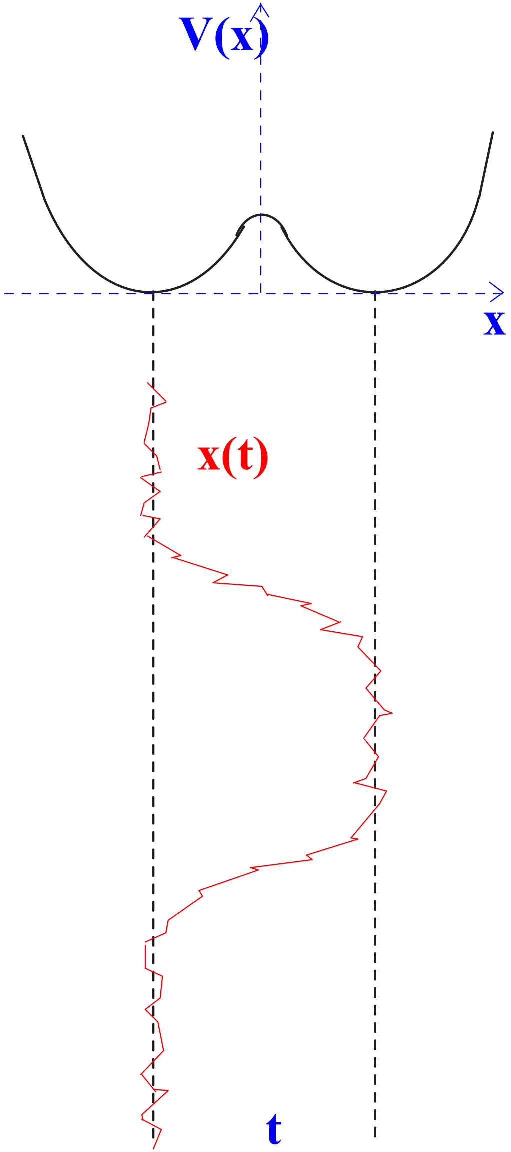
To cut out the ground state: take $t \to -i\tau,\ T \to -iT,\ (T \to \infty)$. Put $x_1 = x_2$ and integrate over it. This is called the partition function: (henceforth $\hbar \to 1$)

$$\begin{aligned}\mathcal{Z} &= \sum_n e^{-E_n T} \\ &= \int Dx(\tau)\, \exp\left\{-\int_0^T d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V\left(x(\tau)\right)\right]\right\}.\end{aligned}$$

In statistical mechanics one implies that $T = \frac{1}{kT^{\mathrm{o}}} = \beta$ where T^{o} is temperature. In particle physics T is the observation (Euclidean) time. If one is interested only in the ground (or vacuum) state E_0 one cuts it out by taking temperature $T^{\mathrm{o}} \to 0$ or observation time $T \to \infty$.

How do typical trajectories $x(\tau)$ look like?


V(x)
x
x(t)
t


V(x)
x
x(t)
t

Text-book example: harmonic oscillator,

$$V(x) = \tfrac{m}{2}\omega_0^2\, x^2.$$

Since $x(t+T) = x(t)$ we can decompose arbitrary trajectory (over which we have to integrate!) in periodic functions,

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \frac{1}{\sqrt{T}} \exp\left(2\pi i\, n \frac{t}{T}\right), \qquad c_{-n} = c_n^*$$

and the path integral will be understood as a product of integrals over all Fourier coefficients,

$$\int Dx(t) = \prod_n \int \frac{dc_n}{\sqrt{2\pi}}.$$

We shall need the ortho-normalization condition,

$$\int_0^T dt \, \frac{1}{\sqrt{T}} \exp\left(2\pi i \, m \frac{t}{T}\right) \frac{1}{\sqrt{T}} \exp\left(-2\pi i \, n \frac{t}{T}\right) = \delta_{m,n}.$$

We get in the exponent of the path integral

$$\frac{m}{2} \int dt \, (\dot{x}^2 + \omega_0^2 \, x^2) = \frac{m}{2} \sum_{n=-\infty}^{\infty} c_{-n} \left[\left(\frac{2\pi n}{T} \right)^2 + \omega_0^2 \right] c_n$$

so that

$$\prod_{n=1} \int \frac{dc_n}{\sqrt{2\pi}} \frac{dc_{-n}}{\sqrt{2\pi}} \exp\left(-\frac{1}{2} \sum_n c_{-n} \, \lambda_n \, c_n \right) = \prod_{n=1} \frac{1}{\lambda_n} \, .$$

Harmonic oscillator's partition function is

$$\mathcal{Z} = \prod_n \frac{1}{\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2} \longrightarrow \prod_{n=1}^{\infty} \frac{\left(\frac{2\pi n}{T}\right)^2}{\left(\frac{2\pi n}{T}\right)^2 + \omega_0^2}$$

$$= \frac{\frac{\omega_0 T}{2}}{\sinh \frac{\omega_0 T}{2}} \left(\times \frac{1}{\omega_0 T}\right) = \frac{1}{e^{\frac{\omega_0 T}{2}} - e^{-\frac{\omega_0 T}{2}}}$$

$$= e^{-\frac{\omega_0 T}{2}} \left(1 + e^{-\omega_0 T} + e^{-2\omega_0 T} + e^{-3\omega_0 T} + \ldots\right)$$

$$= \sum_n e^{-E_n T} \overset{T\to\infty}{=} e^{-\frac{\omega_0 T}{2}}, \qquad E_n = \omega_0 \left(\frac{1}{2} + n\right)$$

[don't forget that in statistical mechanics $\omega_0 \longrightarrow \hbar\omega_0$, $T \longrightarrow \frac{1}{kT^o}$]

Important lesson:

Path Integral = Product of integrals over Fourier coefficients

Quantum mechanics can be called $0+1$-dimensional field theory. In true field theory the 'coordinates' are fields depending on real space coordinates $\mathbf{x}$.

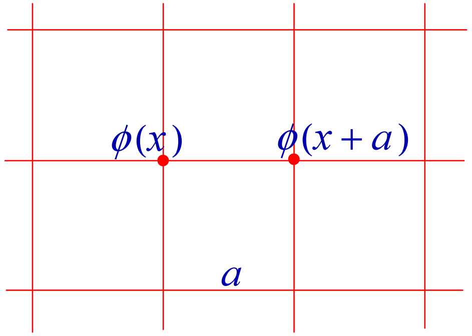
Typical path integral in Quantum Field Theory:

$$\mathcal{Z}=\int D\phi(\mathbf{x},t)\,\exp\left(-S[\phi,\partial\phi,\ldots]\right)\,.$$

For example,

$$S[\phi,\partial\phi,\ldots]=\int_0^T dt\int d^dx\left(\frac{1}{2}\partial_0\phi\partial_0\phi+\frac{1}{2}\partial_i\phi\partial_i\phi+\frac{\lambda}{4}\phi^4\right)\,.$$

It is a field theory in $d+1$ dimensions.



$$\left(\frac{\phi(x+a)-\phi(x)}{a}\right)^2 = \left(\frac{\Delta\phi}{a}\right)^2 \overset{a\to 0}{=} \partial_x\phi\partial_x\phi$$

$$\mathcal{Z} = \prod_n \int d\phi(x_n)\ \exp\left(-a^2\sum_{\text{links}}(\Delta\phi)^2 + a^4\sum_{\text{sites}}\frac{\lambda}{4}\phi^4(x_n)\right)$$

References

R.P. Feynman and A.R. Hibbs, Quantum Mechanics and Path Integrals, McGraw-Hill, N.Y. (Russian translation at http://irodov.nm.ru/books.htm)

Problem

Read chapters 2 and 3 of that book. Find misprints in eqs. (3.62) and (3.64).

Classical solutions. Instantons and solitons.

A very wide and useful class of methods exploited equally fruitfully both in condensed matter and particle physics, are semiclassical methods. They are applicable if, for some reasons, a path integral has saddle point(s) and, in addition, saddle-point calculation is justified by some large (or small) parameter. It happens quite often.

Two cases should be distinguished from the start. Case one: the action has a saddle point. The corresponding solution of the Euler–Lagrange eqns. of motion is generically called an instanton ('t Hooft's term):

$$\mathcal{Z} = \int D\phi(x,t)\,\exp\left\{-\int dt\int d^d x\,\mathcal{L}(\phi,\dot{\phi},\partial\phi)\right\}$$

$$\phi^{\rm class}(x,t):\qquad -\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\phi}}-\frac{\partial}{\partial_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}+\frac{\partial\mathcal{L}}{\partial\phi}=0.$$

Case two: the energy functional has a saddle point

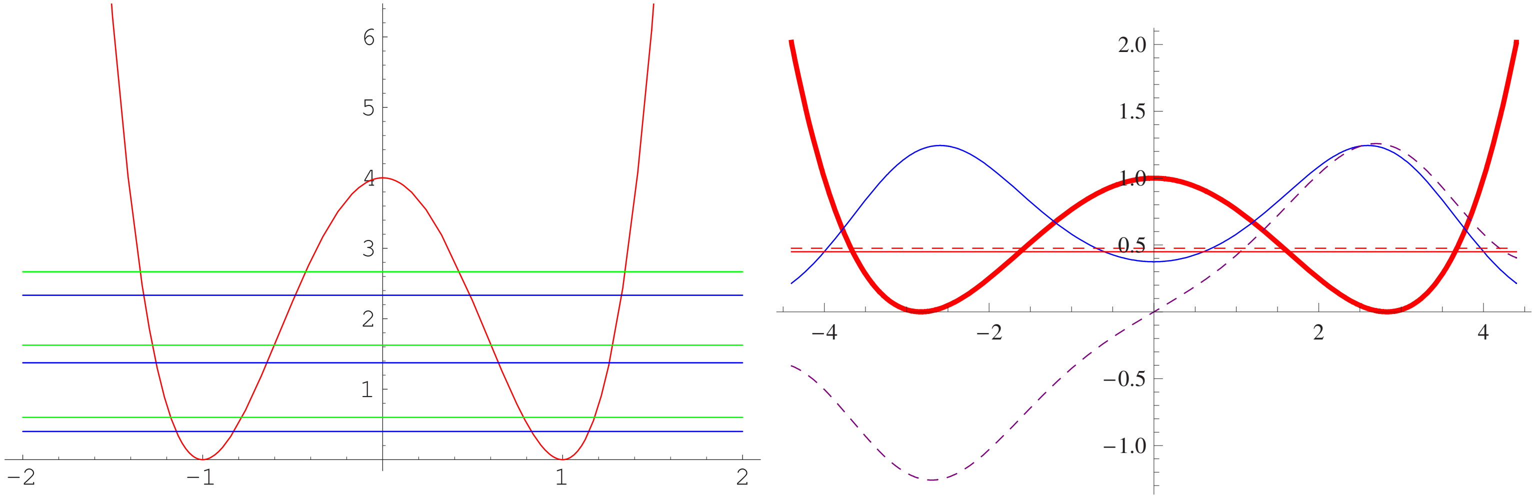
$$\phi^{\mathrm{class}}(x)\ : \qquad -\frac{\partial}{\partial_i}\frac{\partial\mathcal{L}}{\partial(\partial_i\phi)}+\frac{\partial\mathcal{L}}{\partial\phi}=0.$$

Then the solution is generically called a soliton.

- instanton is a saddle point of the action
- soliton is a saddle point of the energy

An instanton in d dimensions is often a soliton in $d+1$ dimensions. Instantons are more easy to deal with, so let us start from instantons.

Quantum mechanics, double-well potential



All levels are split into two. One can calculate level splitting in two ways:

- Solving Schrödinger eqn
- From instantons [A. Polyakov (1974)]

Schrödinger eqn:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\,\psi = \epsilon\,\psi,$$

$$V(x) \quad = \quad V_0\left(\frac{x^2}{x_0^2} - 1\right)^2, \quad \left.\frac{d^2V}{dx^2}\right|_{x=\pm x_0} = \frac{8V_0}{x_0^2} = m\omega_0^2.$$

We shall use units $\hbar = m = \omega_0 = 1 \implies V_0 = (1/8)\,x_0^2$. Neglecting the influence of the other well the Schrödinger Eqn becomes

$$-\tfrac{1}{2}\frac{d^2\psi}{dx^2} + \tfrac{1}{2}(x \pm x_0)^2\,\psi = \epsilon\,\psi, \qquad \epsilon_n = n + \tfrac{1}{2}.$$

Approximate ground-state wave functions for the double-well potential:

$$\psi_{\mathrm{s,a}}(x) = \frac{\psi(x - x_0) \pm \psi(x + x_0)}{\sqrt{2}}, \qquad \psi(x) \sim \exp(-\tfrac{1}{2}x^2).$$

Let us solve the double-well Schrödinger Eqn more accurately. Near $x = -x_0$

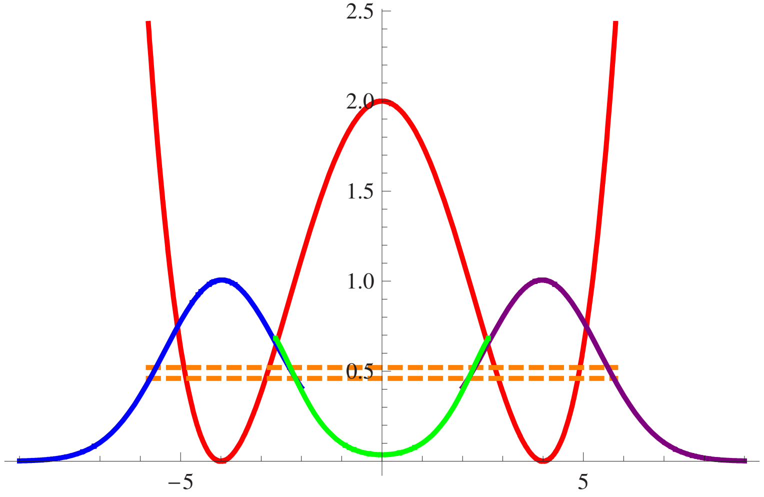
$$\psi_{\rm I}(x) = c_{\rm I}\, D_{\epsilon-\frac{1}{2}}(-(x+x_0))$$

where $D_\nu(x)$ is the *parabolic cylinder* function. It decays at $x \longrightarrow -\infty$ but grows at $x \longrightarrow +\infty$. Near $x = x_0$

$$\psi_{\rm II}(x) = c_{\rm II}\, D_{\epsilon-\frac{1}{2}}((x-x_0))$$

It decays at $x \longrightarrow +\infty$ but grows at $x \longrightarrow -\infty$. Under the barrier one writes the WKB semiclassical wave function being a linear combination of rising and falling functions,

$$\begin{aligned} \psi_{\rm III}(x) &= \frac{c}{\sqrt{p(x)}}\left[\exp\left(\int_0^x p\,dx\right) \pm \exp\left(-\int_0^x p\,dx\right)\right], \\ p(x) &= \sqrt{2\,(V(x)-\epsilon)} \end{aligned}$$



Now one has to find the coefficients $c_{\mathrm{I,II}}$ from c by matching the wave functions and their derivatives near the two regions where the wave functions 'dive' under the barrier. That gives the eigenvalues ϵ_n:

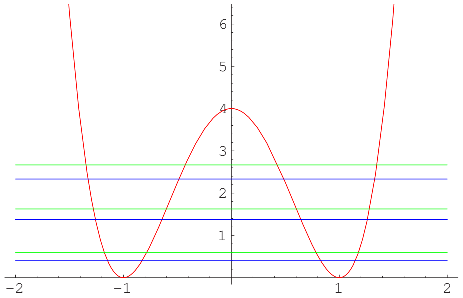
$$\epsilon_n^{\pm} \quad = \quad n + \tfrac{1}{2} \pm \tfrac{1}{2}\Delta_n + \left(\text{series in } \frac{1}{V_0}\right),$$

$$\Delta_n \quad = \quad \frac{4}{n!}\sqrt{\frac{8V_0}{\pi}}(64V_0)^n \exp\left(-\frac{16}{3}V_0\right)(1+O(1/V_0)).$$

Problem: Do it, exploiting the asymptotics of the parabolic cylinder functions which you can find e.g. in *Mathematica*

Conclusions:

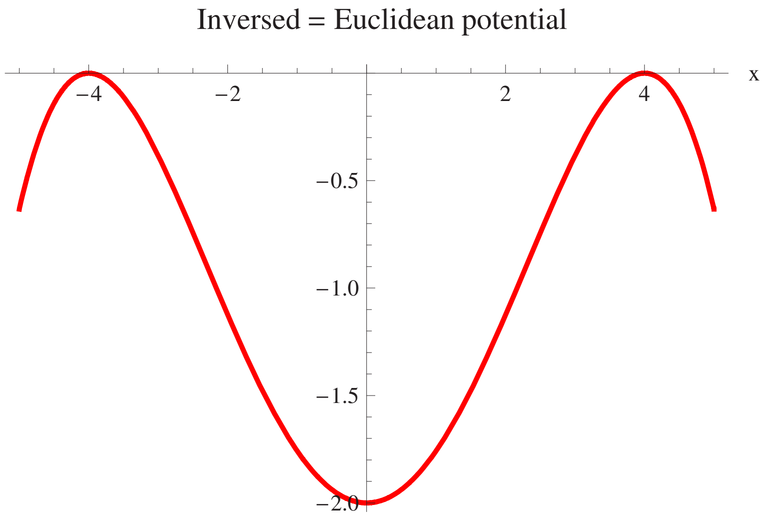
- Splittings are exponentially small in the barrier hight V_0
- Splittings increase with the level number n.



Instanton

We shall now derive the same result for level splittings using quantum field theory. Recall Feynman's path integral:

$$\begin{aligned}\langle x_0|e^{-HT}|-x_0\rangle &= \int Dx(t)\,\exp\left(-\int_0^T dt\left[\frac{\dot{x}^2}{2}+V_E(x)\right]\right),\\ V_E(x) &= -V(x).\end{aligned}$$



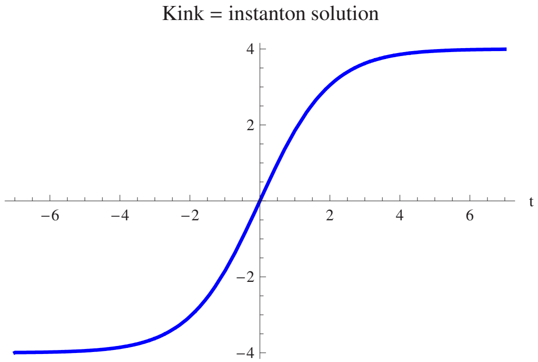
Classical equation of motion

$$-\ddot{x} - V_E'(x) = 0 \quad \text{or}$$

$$\dot{x} \quad = \quad \sqrt{2[E - V_E(x)]} \quad \overset{E=0}{=} \quad \frac{\sqrt{2V_0}}{x_0^2}(x^2 - x_0^2), \quad 8V_0 = x_0^2.$$

Its solution with zero Euclidean 'energy' is a kink:

$$x(t) = x_0 \, \tanh \frac{t}{2}.$$



It is the instanton of the $0+1$ dimensional field theory. The action along the tunneling trajectory is

$$\begin{aligned} S &= \int dt \left[\frac{\dot{x}^2}{2} + V(x)\right] = \int dt\, \dot{x}^2 = \frac{x_0^2}{4} \int_{-\infty}^{+\infty} \frac{dt}{\cosh^2 \frac{t}{2}} \\ &= \frac{2}{3} x_0^2 = \frac{16}{3} V_0 \gg 1 \end{aligned}$$

Tunneling amplitude

$$\mathcal{A} \sim \exp(-S) = \exp(-\frac{16}{3} V_0) \ll 1.$$

Gives correctly the exponential suppression factor for the level splittings!
How to find the pre-exponential factors in the level splittings?

$\Longrightarrow$ one needs to take into account quantum fluctuations about the instanton,
here: fluctuations of the trajectory about the kink!